Lecture 6: One-sided limits

Objectives:

(6.1) Estimate and evaluate one-sided limits.

(6.2) Use one-sided limits to justify that a limit does not exist

One-sided limits

We saw earlier that a limit fails to exist when the limit from the right is not equal to the limit from the left. It is time for us to formalize the idea of a one-sided limit.

If f is defined on an interval of the form (a, c), then the limit of f as x approaches c from the left (i.e. from values less than c) is denoted by

$$\lim_{x \to c^-} f(x).$$

Similarly, if f is defined on an interval of the form (c, b), then the limit of f as x approaches c from the right (i.e. from values greater than c) is denoted by

$$\lim_{x \to c^+} f(x).$$

Of course one-sided limits may exist at a point when the regular two-sided limit does not. For example,

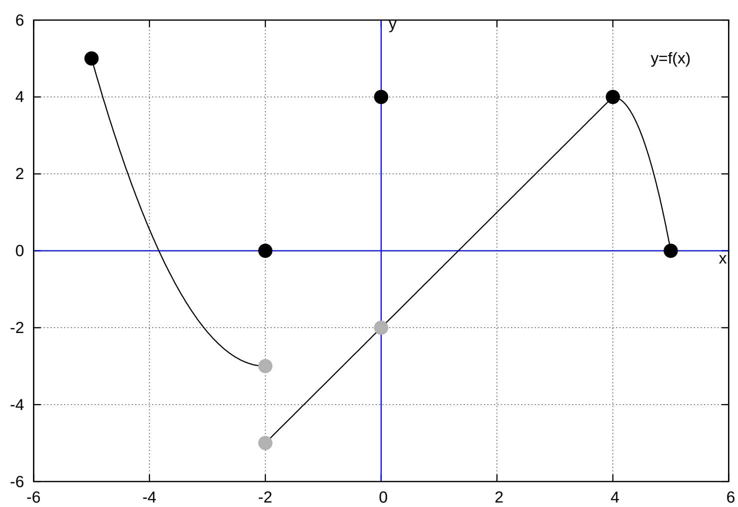
$$\lim_{x \to 0} \sqrt{x} \text{ DNE}$$

because $\sqrt{x}$ is not defined on both sides of $x = 0$. However, since $\sqrt{x}$ is defined to the right of $x = 0$, we have no problem with a limit from the right:

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Example 1 The graph of $y = f(x)$ is shown below. Estimate each of the following for $c = -2$, $c = 0$, and $c = 4$.

$$\lim_{x \to c^-} f(x), \qquad \lim_{x \to c^+} f(x), \qquad \lim_{x \to c} f(x), \qquad f(c)$$



Solution omitted.

The relationship between one- and two-sided limits

It should be intuitively clear that if a normal, two-sided limit exists at a point, then both one-sided limits exist and are equal. This idea is important enough to state as a theorem.

Theorem 1 — One- and two-sided limits
Suppose f is defined on an open interval containing c, except possibly at c. Then

$$\lim_{x \to c} f(x) = L$$

if and only if

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

One consequence of this theorem is that we can evaluate one-sided limits using the same techniques we used to evaluate two-sided limits.

Example 2 Evaluate the limit: $\lim\limits_{x \to 3^+} x^2 \sin^2 \pi x$.

The limit at $x = 3$ exists and can be determined by direct substitution. Therefore, both the left and right limits exist and can also be determined by direct substitution.

$$\lim_{x \to 3^+} x^2 \sin^2 \pi x = 3^2 \sin^2(3\pi) = (9)(0) = 0.$$

Example 3 Evaluate the limit: $\lim\limits_{h \to 1^-} \dfrac{1-h}{\sqrt{1-h}}$.

First notice that $\sqrt{1-h}$ is only defined to the left of $h = 1$, so we can only consider the limit from the left. Direct substitution of $h = 1$ yields the indeterminate form $0/0$. As usual, we conclude nothing without doing more work.

$$\lim_{h \to 1^-} \frac{1-h}{\sqrt{1-h}} = \lim_{h \to 1^-} \frac{\sqrt{1-h}\cancel{\sqrt{1-h}}}{\cancel{\sqrt{1-h}}} = \lim_{h \to 1^-} \sqrt{1-h} = \sqrt{0} = 0.$$

Example 4 Consider the following piece-wise defined function:

$$g(x) = \begin{cases} 2x + 5, & x < 3 \\ x^3 - 8x + 1, & x > 3 \end{cases}$$

1. Find the limit: $\lim\limits_{x \to 3^-} g(x)$

   The function g is defined in two pieces with the breakpoint at $x = 3$, which happens to be the limit point. In order to evaluate the limit from the left at $x = 3$, we must use the piece of the function that defines g to the left of $x = 3$. Using that piece, direct substitution gives

   $$\lim_{x \to 3^-} g(x) = \lim_{x \to 3} (2x + 5) = 11.$$

2. Find the limit: $\lim\limits_{x \to 3^+} g(x)$

   Reasoning as above,

   $$\lim_{x \to 3^+} g(x) = \lim_{x \to 3} (x^3 - 8x + 1) = 4.$$

3. Find the limit: $\lim\limits_{x \to 3} g(x)$

Since the limit from the left of $x = 3$ does not equal the limit from the right, the two-sided limit at $x = 3$ does not exist.

4. Find the limit: $\lim_{x \to 0} g(x)$

Since the limit point at $x = 0$ is to the left of the function's breakpoint at $x = 3$, we need only consider the piece of the function that defines g to the left of 3.

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} (2x + 5) = 5.$$

In order to evaluate one-sided limits involving absolute value, we will often have to rewrite the expression by using the definition of the absolute value function:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Example 5 Evaluate $\lim_{x \to 0^-} \frac{|x|}{x}$ and $\lim_{x \to 0^+} \frac{|x|}{x}$.

In each case, direct substitution yields the indeterminate form 0/0. We use the piece-wise definition of $|x|$.

To the left of $x = 0$, we have

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0} \frac{-x}{x} = \lim_{x \to 0} -1 = -1.$$

To the right of $x = 0$, we have

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} 1 = 1.$$

Example 6 Evaluate $\lim_{x \to 2^-} \frac{|x-2|(x^2+x)}{x-2}$ and $\lim_{x \to 2^+} \frac{|x-2|(x^2+x)}{x-2}$.

In each case, direct substitution yields the indeterminate form 0/0. We use the piece-wise definition of $|x|$.

To the left of $x = 2$, we have

$$\lim_{x \to 2^-} \frac{|x-2|(x^2+x)}{x-2} = \lim_{x \to 2} \frac{-(x-2)(x^2+x)}{x-2} = \lim_{x \to 2} -(x^2+x) = -6.$$

To the right of $x = 2$, we have

$$\lim_{x \to 2^+} \frac{|x-2|(x^2+x)}{x-2} = \lim_{x \to 2} \frac{(x-2)(x^2+x)}{x-2} = \lim_{x \to 2} (x^2+x) = 6.$$

Example 7 Sketch the graph of a function f such that

- $\lim_{x \to 1^-} f(x) = 2$
- $\lim_{x \to 1^+} f(x) = 1$
- $f(1) = 0$
- $\lim_{x \to -1^+} f(x) = -2$
- $\lim_{x \to -1} f(x)$ exists

Solution omitted. Answers vary.