Lecture 8: Infinite limits

Objectives:

(8.1) Determine one-sided and two-sided infinite limits.

(8.2) Find and sketch the vertical asymptotes of the graph of a function.

Determining infinite limits

If function values grow without bound as a point is approached, the limit at that point does not exist. This was the second way that we saw a limit could fail to exist. Because this is such a special way that a limit fails to exist, we will be more specific about this type of failure. When appropriate, we will say that the limit is infinite. For example, as x approaches zero (from either side), the values of $1/x^2$ are always positive, and they grow without bound. We write

$$\lim_{x\to 0} \frac{1}{x^2} = +\infty.$$

Notice that direct substitution into the limit above yields the form $1/0$. In order to recognize an infinite limit, we must be able to recognize when a function is growing without bound. The simplest way to do this is to use direct substitution and look for the form $(k \neq 0)/0$. Remember that $0/0$ is an indeterminate form, but the form "nonzero over zero" always indicates that values are growing without bound. Before we can assign an infinite limit, however, we must determine exactly how the values are growing. Not every $(k \neq 0)/0$ form can be assigned an infinite limit. For example,

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty, \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty, \qquad \lim_{x\to 0} \frac{1}{x} \text{ DNE.}$$

In order to resolve a $(k \neq 0)/0$ form, we must assign the left and right infinite limits by determining the signs of the function values just to the left and right of the limit point. When we say a limit is infinite, we are not trying to say that the limit exists, we are just being specific about how it fails to exist. If the behavior of the function is different on opposite sides of the limit point, we will continue to say the limit does not exist. The following examples illustrate our procedure for assigning infinite limits.

Example 1 Evaluate the limit. Use $+\infty$ or $-\infty$ if appropriate. $\displaystyle\lim_{x\to 5} \frac{3x}{x-5}$

Direct substitution yields the form $15/0$. We expect infinite limits, so we look carefully at both sides of $x = 5$. Just to the left of $x = 5$, the numerator will be positive and the denominator will be negative. Therefore the function values will be negative and the left-hand limit is $-\infty$:

$$\lim_{x\to 5^-} \frac{3x}{x-5} = -\infty.$$

Just to the right of $x = 5$, both the numerator and denominator will be negative, and therefore the right-hand limit is $+\infty$:

$$\lim_{x\to 5^+} \frac{3x}{x-5} = +\infty.$$

It follows that

$$\lim_{x\to 5} \frac{3x}{x-5} \text{ DNE.}$$

Example 2 Evaluate the limit. Use $+\infty$ or $-\infty$ if appropriate. $\displaystyle\lim_{x\to 6} \frac{2x-12}{x^2-12x+36}$

Direct substitution yields the form $0/0$. We cannot draw any conclusions without further analysis.

$$\lim_{x\to 6} \frac{2x-12}{x^2-12x+36} = \lim_{x\to 6} \frac{2\cancel{(x-6)}}{\cancel{(x-6)}(x-6)} = \lim_{x\to 6} \frac{2}{x-6}$$

At this point, direct substitution gives $2/0$. We expect infinite limits, and we look carefully at both sides of $x = 6$. From the left, the numerator is positive and the denominator is negative. Therefore the function values are negative.

$$\lim_{x \to 6^-} \frac{2}{x-6} = -\infty$$

From the right of $x = 6$, both the numerator and denominator are positive.

$$\lim_{x \to 6^-} \frac{2}{x-6} = +\infty$$

Our overall conclusion is that the original limit does not exist.

Example 3 Evaluate the limit. Use $+\infty$ or $-\infty$ if appropriate. $\displaystyle\lim_{x \to -1} \frac{x-3}{(x+1)^2}$

Direct substitution yields the form $-4/0$. We expect infinite limits, and we look carefully at both sides of $x = -1$. Just to the left of $x = -1$, the numerator is negative and the denominator is positive.

$$\lim_{x \to -1^-} \frac{x-3}{(x+1)^2} = -\infty$$

Just to the right of $x = -1$, the numerator is negative and the denominator is positive.

$$\lim_{x \to -1^+} \frac{x-3}{(x+1)^2} = -\infty$$

We conclude that the two-sided limit is $-\infty$.

Example 4 Evaluate the limit. Use $+\infty$ or $-\infty$ if appropriate. $\displaystyle\lim_{x \to \pi/2^-} \tan x$

We could simply use our knowledge of the tangent function to assign this limit, but instead we will take the same approach as above. After rewriting $\tan x = \frac{\sin x}{\cos x}$, we see that direct substitution gives $1/0$. We expect an infinite limit, and we look carefully at the left of $\pi/2$. Just to the left, $\sin x$ is positive, $\cos x$ is positive, and therefore the values of $\tan x$ are positive.

$$\lim_{x \to \pi/2^-} \tan x = +\infty$$

Vertical asymptotes

At the places where a function has a one- or two-sided infinite limit, the graph has a vertical asymptote. We are familiar with vertical asymptotes from our earlier algebra courses, but the limit concept provides us with a simple definition.

Definition of vertical asymptote
The line $x = c$ is a vertical asymptote of the graph of f if any one of the following is true:

$$\lim_{x \to c^-} f(x) = -\infty, \qquad \lim_{x \to c^-} f(x) = +\infty, \qquad \lim_{x \to c^+} f(x) = -\infty, \qquad \lim_{x \to c^+} f(x) = +\infty$$

To find the vertical asymptotes of the graph of a function f, we must determine the points at which f has a one- or two-sided infinite limit. Of course, we can start our search by looking for points at which f has a zero denominator. As we saw in example 2, however, a zero denominator does not necessarily signal an infinite limit.

The following examples will help dispell some common myths about vertical asymptotes.

Example 5 Find all vertical asymptotes of the graph of $g(x) = \dfrac{x-1}{x^2-4x+3}$.

In order to find the graph's vertical asymptotes, we must determine if and where g has any infinite limits. Infinite limits will occur at points where the limit of g has the form $(k \neq 0)/0$. Let's start by identifying the locations at which the denominator is zero.

$$x^2 - 4x + 3 = (x-1)(x-3) = 0 \quad \Longrightarrow \quad x = 1 \text{ or } x = 3$$

$x = 1$ and $x = 3$ are candidates for vertical asymptotes, but we must check the limits.

$$\lim_{x \to 1^-} \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x-3)} = -\frac{1}{2} \qquad \lim_{x \to 1^+} \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x-3)} = -\frac{1}{2}$$

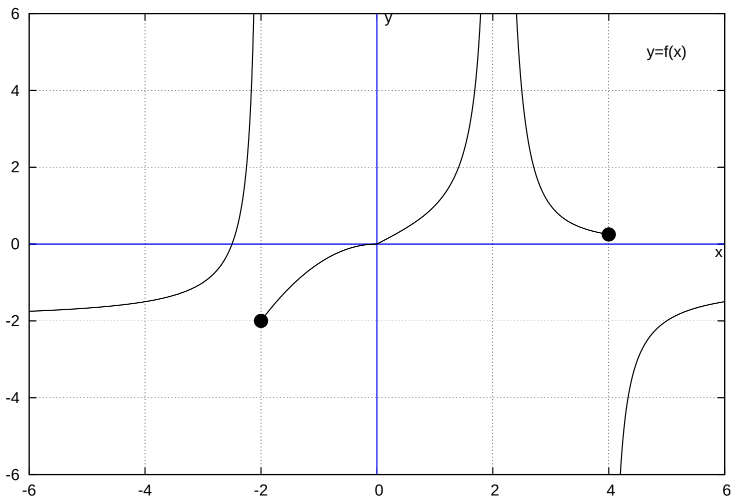
There is no infinite limit at $x = 1$, so there is no vertical asymptote at $x = 1$.

$$\lim_{x \to 3^-} \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x-3)} = -\infty \qquad \lim_{x \to 3^+} \frac{\cancel{(x-1)}}{\cancel{(x-1)}(x-3)} = +\infty$$

The one-sided infinite limits at $x = 3$ indicate that $x = 3$ is a vertical asymptote.

Example 6 Is it possible for a function to be defined at a point at which the graph has a vertical asymptote?

Because of our experience with rational functions (examples 1, 2, 3, and 5), we are inclined to answer "no". However, if we look back at the definition of vertical asymptote, we see nothing to indicate that a function cannot be defined where its graph has a vertical asymptote. Indeed, the graph shown below has three vertical asymptotes: $x = -2$, $x = 2$, and $x = 4$. Also notice that $f(-2)$ and $f(4)$ exist, while $f(2)$ apparently does not.



Example 7 Find all vertical asymptotes of the graph of $f(x) = \dfrac{x}{\sin x}$.

Since $\sin x = 0$ when x is any multiple of π, candidates for vertical asymptotes are $x = k\pi$, for any integer k. For each nonzero integer k, direct substitution into the expression for f gives the form "nonzero over zero." These must give rise to vertical asymptotes. If on the other hand, $k = 0$, then we are looking at $x = 0$, and the limit is not infinite:

$$\lim_{x \to 0} \frac{x}{\sin x} = 1.$$

When all is said and done, we see that the graph has a vertical asymptote at any nonzero multiple of π.