Section 7.4 - Area and Arc Length in Polar Coordinates

In this section, we will look at some applications of calculus to polar curves.

Theorem 1 (Area of polar region)

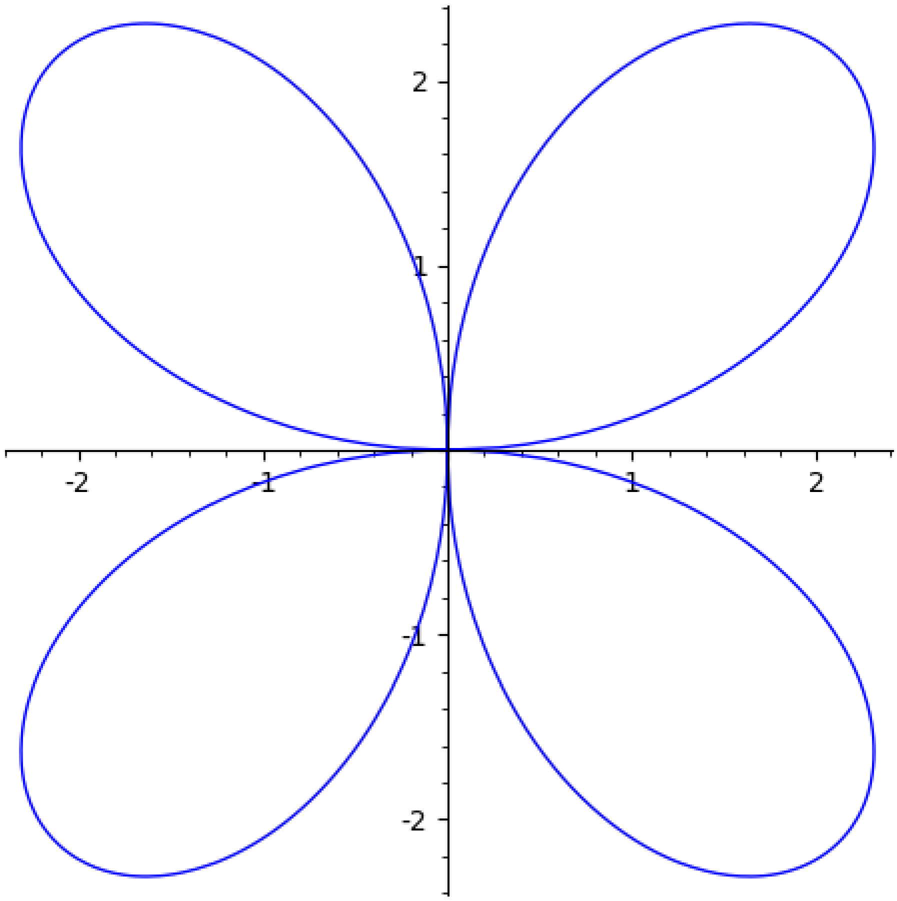
Suppose f is continuous and nonnegative on the interval $\alpha \leq \theta \leq \beta$ with $0 \leq \beta - \alpha \leq 2\pi$. The area of the region bounded by the polar curve $r = f(\theta)$ and the radial lines $\theta = \alpha$ and $\theta = \beta$ is given by

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta.$$

Example 1

Find the area of one petal of the rose curve defined by $r = 3\sin(2\theta)$.

The graph of the rose curve is shown below.



At $\theta = 0$, we have $r = 0$. The next value of θ for which $r = 0$ is $\theta = \pi/2$. Therefore one complete petal is traced out as θ varies from 0 to $\pi/2$. The area of the petal is given by

$$A = \frac{1}{2}\int_{0}^{\pi/2} 9\sin^2(2\theta) \, d\theta.$$

This integral can be evaluated by using the appropriate power reducing formula. You should work through the details for practice, but for now, let's use a computer algebra system. The following SageMath code

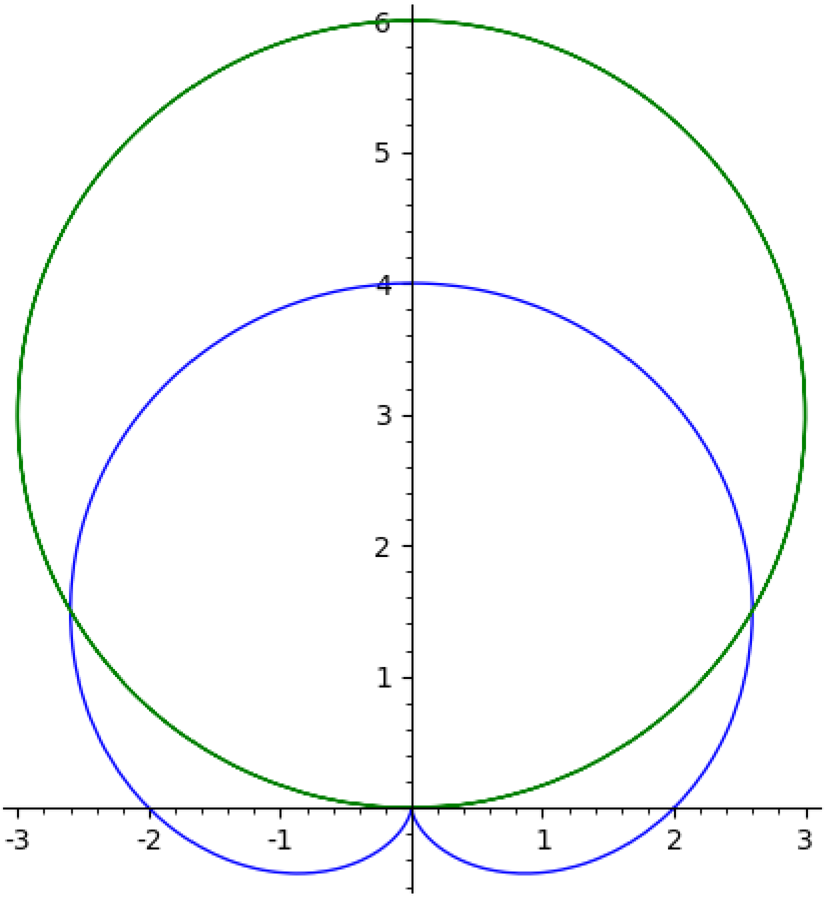
```
(1/2)*integrate(9*sin(2*x)^2,x,0,pi/2)
```

gives $9\pi/8$. ♢

Example 2

Find the area outside the graph of $r = 2 + 2\sin\theta$ (called a *cardioid*) and inside the circle described by $r = 6\sin\theta$.

The graphs of the curves are shown below.



By setting $2 + 2\sin\theta = 6\sin\theta$, we can find the intersection points.

$$\begin{aligned}
2 + 2\sin\theta &= 6\sin\theta \\
2 &= 4\sin\theta \\
\sin\theta &= \frac{1}{2} \\
\theta &= \frac{\pi}{6}, \frac{5\pi}{6}
\end{aligned}$$

The area of the region between the curves is the area inside the outer curve minus the area inside the inner curve:

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (6\sin\theta)^2\, d\theta - \frac{1}{2}\int_{\pi/6}^{5\pi/6} (2 + 2\sin\theta)^2\, d\theta.$$

Once again, this is a great integral to evaluate for practice, but right now, we'll use technology. The following SageMath code

```
(1/2)*integrate((6*sin(x))^2-(2+2*sin(x))^2,x,pi/6,5*pi/6)
```

gives 4π. ♢

Theorem 2 (Arc length of a polar curve)

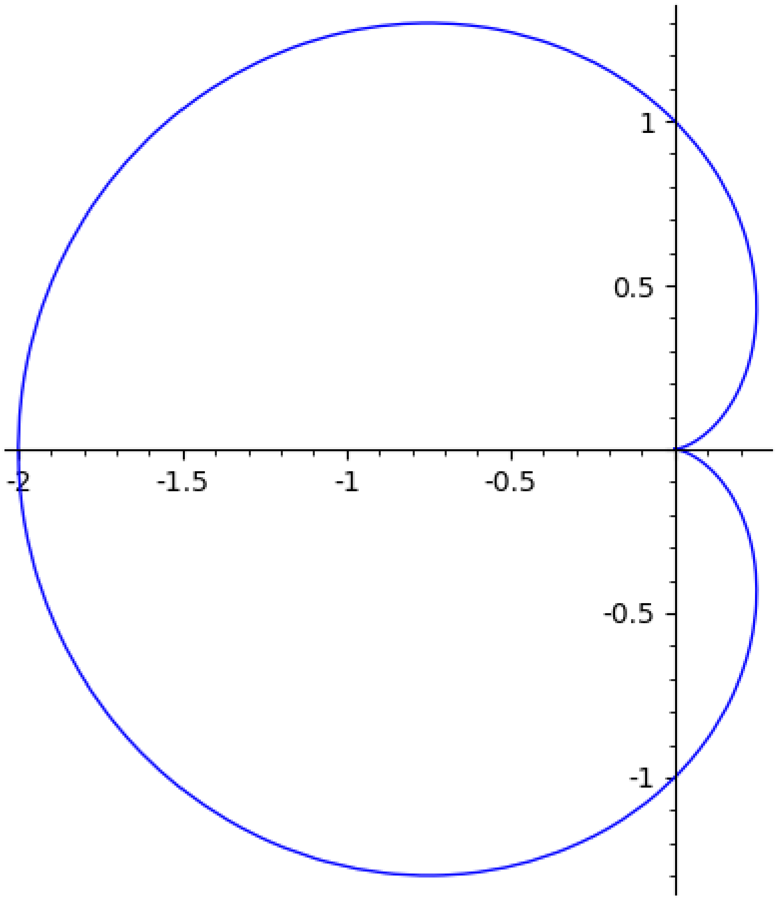
Let f be a function whose derivative is continuous on the interval $\alpha \leq \theta \leq \beta$. The length of the graph of the polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is given by

$$L = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta.$$

Example 3

Find the length of the graph of the polar curve described by $r = 1 - \cos\theta$.

The graph of the curve is shown below.



At $\theta = 0$, we have $r = 0$. The next value of θ for which $r = 0$ is $\theta = 2\pi$. Therefore the complete curve is traced out once as θ varies from 0 to 2π. The length of the curve is

$$L = \int_0^{2\pi} \sqrt{[1 - \cos\theta]^2 + [\sin\theta]^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = 8,$$

where the TI-84 was used to approximate the value of the integral. ♢