Section 5.6 - Ratio and Root Tests

There are literally hundreds of tests for the convergence or divergence of series. We've studied a few of the most common tests. We'll close chapter 5 by studying two more tests, which are easy to use, apply to very general series, and are the prefect tests of choice for certain common situations. A summary of the tests we have studied can be found at `http://stevekifowit.com/sheets/c05.pdf`. There is also a similar summary table in the textbook.

Theorem 1 (Ratio test)

Suppose $\sum a_n$ is a series with nonzero terms, and let $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

1. If $0 \leq \rho < 1$, then $\sum a_n$ converges absolutely.
2. If $\rho > 1$ or $\rho = \infty$, then $\sum a_n$ diverges.
3. If $\rho = 1$, the test does not provide any information.

Example 1

Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges or diverges.

Before we start working on this example, recall that $n!$ (read "n factorial") means $1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1) \cdot n$, and that we assign the value of 1 to $0!$ So, the first few terms of this series are

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} = \frac{1}{1} + \frac{2}{1} + \frac{4}{2} + \frac{8}{6} + \frac{16}{24} + \frac{32}{120} + \cdots.$$

The ratio test is a often very useful for series involving factorials. Let's try it...

$$\rho = \lim_{n\to\infty} \left|\frac{2^{n+1}/(n+1)!}{2^n/n!}\right| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n},$$

where we wrote the division as multiplication by the reciprocal, and we lost the absolute values (since all the terms are positive). Now notice that $(n+1)! = (n+1) \cdot n!$, so that

$$\frac{n!}{(n+1)!} = \frac{1}{n+1}.$$

It follows then that

$$\rho = \lim_{n\to\infty} \frac{2}{n+1} = 0.$$

By the ratio test, the series converges. In this example, there is no need to say "converges absolutely" because the terms are all positive anyway. $\diamondsuit$

Example 2

The ratio test is inconclusive when applied to the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\rho = \lim_{n\to\infty}\left|\frac{1}{n+1}\cdot\frac{n}{1}\right| = 1.$$

Nonetheless, we already know that the harmonic series diverges. $\diamondsuit$

Example 3

Determine whether the series $\displaystyle\sum_{n=0}^{\infty}\frac{n^2 2^{n+1}}{3^n}$ converges or diverges.

This is another great example for the ratio test. First compute $a_{n+1} = \dfrac{(n+1)^2 2^{n+2}}{3^{n+1}}$, so that

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2 2^{n+2}}{3^{n+1}}\cdot\frac{3^n}{n^2 2^{n+1}}\right| = \lim_{n\to\infty}\frac{2(n+1)^2}{3n^2}$$

Instead of trying to use L'Hopital's rule to find the limit, let's just divide both the numerator and denominator by n^2. This is a technique you learned in Calculus 1 for evaluating limits at infinity.

$$\rho = \lim_{n\to\infty}\frac{2(n+1)^2}{3n^2}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{2(1+\frac{1}{n})^2}{3} = \frac{2}{3}$$

Since $\rho < 1$, the series converges. Once again, since all terms are positive, there is no need to distinguish between convergence and absolute convergence. $\diamondsuit$

Theorem 2 (Root test)

Consider the series $\sum a_n$ and let $\rho = \lim\limits_{n\to\infty}\sqrt[n]{|a_n|}$.

1. If $0 \le \rho < 1$, then $\sum a_n$ converges absolutely.
2. If $\rho > 1$ or $\rho = \infty$, then $\sum a_n$ diverges.
3. If $\rho = 1$, the test does not provide any information.

Example 4

Determine whether $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n e^{2n}}{n^n}$ converges or diverges.

Because the terms are nth powers, $a_n = \left(\dfrac{-1\cdot e^2}{n}\right)^n$, this is a great example for the root test.

$$\rho = \lim_{n\to\infty}\sqrt[n]{\left|\frac{-1\cdot e^2}{n}\right|^n} = \lim_{n\to\infty}\frac{e^2}{n} = 0$$

By the root test, the series converges absolutely. $\diamondsuit$

Example 5

Determine whether $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{e}+\frac{1}{n}\right)^n$ converges or diverges.

The terms are powers of n, so the root test seems like a good choice. Since the terms are all non-negative, we can ignore the absolute value in the root test.

$$\rho = \lim_{n\to\infty}\sqrt[n]{\left(\frac{1}{e}+\frac{1}{n}\right)^n} = \lim_{n\to\infty}\left(\frac{1}{e}+\frac{1}{n}\right) = \frac{1}{e} \approx 0.36788$$

The series converges by the root test. Once again, since all terms are positive, there is no need to distinguish between convergence and absolute convergence. $\diamondsuit$

Final comments on chapter 5.

1. Determining whether a series converges or diverges can be very difficult. These kinds of problems require lots of practice.
2. Given a series to test, it is often not clear where to even begin. For general guidelines and advice, read the discussion at the end of section 5.6.