CHAPTER 5 - SEQUENCES AND SERIES

CHAPTER 5 IS DIFFICULT BECAUSE THE MATERIAL IS PROBABLY BRAND NEW FOR MOST OF YOU.

ASK LOTS OF QUESTIONS!

SECTION 5.1 SEQUENCES

A **sequence** IS AN ORDERED LIST OF NUMBERS:

$$a_1, a_2, a_3, a_4, \ldots, a_n, \ldots,$$

WHERE EACH INDEXED SYMBOL a_n DENOTES A REAL NUMBER.

EACH TERM OF THE SEQUENCE MAY BE DEFINED EXPLICITLY BY SOME FORMULA: $a_n = f(n)$ WHERE f IS SOME FUNCTION DEFINED OVER THE POSITIVE INTEGERS. WHEN WE WRITE THE TERMS OF A SEQUENCE, WE USUALLY PUT THEM IN BRACKETS.

Ex 1 SOME EXAMPLES

a) $\{1, 4, 9, 16, 25, 36, \ldots\} = \{n^2\}_{n=1}^{\infty}$ IN THIS CASE, $a_n = n^2$

b) $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\} = \{\frac{n-1}{n}\}_{n=1}^{\infty}$ IN THIS CASE, $a_n = \frac{n-1}{n}$

c) $\{-1, 1, -1, 1, -1, \ldots\} = \{(-1)^n\}_{n=1}^{\infty}$ IN THIS CASE, $a_n = (-1)^n$

Ex 2 FIND A FORMULA FOR THE n^{TH} TERM.

a) $\left\{ \frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \frac{243}{16}, \dots \right\}$

$$a_n = \frac{3^n}{3n+1} \quad \text{FOR } n = 1, 2, 3, \dots$$

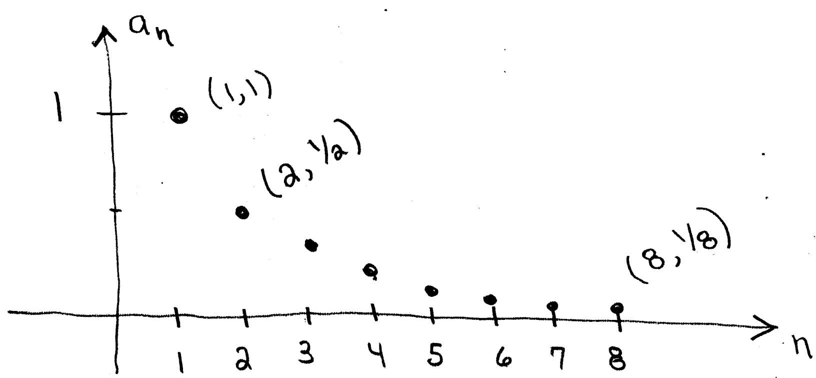
b) $\left\{ -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \dots \right\}$

$$a_n = \frac{(-1)^n n}{n+1} \quad \text{FOR } n = 1, 2, 3, \dots$$

THE <u>GRAPH OF THE SEQUENCE</u> $\{a_n\}_{n=1}^{\infty}$ IS THE SET OF ALL PLOTTED POINTS OF THE FORM (n, a_n).

Ex 3 SKETCH THE GRAPH OF THE SEQUENCE $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

n	a_n
1	1
2	$1/2$
3	$1/3$
4	$1/4$
$\vdots$	$\vdots$



SOME SEQUENCES ARE RECURSIVELY DEFINED AND THEY MAY NOT HAVE EASY-TO-FIND EXPLICIT FORMULAS..

Ex 4 WRITE THE FIRST FIVE TERMS OF EACH RECURSIVE SEQUENCE.

a) $a_0 = 1$ AND $a_n = n \cdot a_{n-1}$ FOR $n = 1, 2, 3, \ldots$

$a_0 = 1, \quad a_1 = 1 \cdot a_0 = 1, \quad a_2 = 2 \cdot a_1 = 2,$

$a_3 = 3 \cdot a_2 = 6, \quad a_4 = 4 \cdot a_3 = 24$

$\{a_n\}_{n=0}^{\infty} = \{1, 1, 2, 6, 24, \ldots\}$

THESE ARE CALLED FACTORIALS.

b) $a_1 = 1, \; a_2 = 1, \; a_{n+1} = a_n + a_{n-1}$ FOR $n = 2, 3, 4, \ldots$

$a_1 = 1$

$a_2 = 1$

$a_3 = a_2 + a_1 = 2$

$a_4 = a_3 + a_2 = 3$

$a_5 = a_4 + a_3 = 5$

$\{a_n\}_{n=1}^{\infty} = \{1, 1, 2, 3, 5, \ldots\}$

THESE ARE THE FIBONACCI NUMBERS.

DEFINITION: THE SEQUENCE $\{a_n\}$ <u>CONVERGES</u> TO THE NUMBER L IF FOR ANY POSITIVE NUMBER ϵ THERE IS A CORRESPONDING POSITIVE INTEGER N SUCH THAT

$$|a_n - L| < \epsilon \text{ WHENEVER } n > N.$$

IF $\{a_n\}$ CONVERGES TO L, WE WRITE

$$\lim_{n\to\infty} a_n = L$$

AND WE SAY L IS THE LIMIT.

IF THERE IS NO LIMIT, WE SAY THE SEQUENCE DOES NOT CONVERGE OR DIVERGES.

WE NORMALLY DON'T USE THAT FORMAL DEFINITION TO CHECK FOR CONVERGENCE. INSTEAD WE USE...

THEOREM: SUPPOSE $f(x)$ IS DEFINED FOR ALL X GREATER THAN SOME FIXED POSITIVE INTEGER AND THAT THE SEQUENCE a_n MATCHES f SO THAT $f(n) = a_n$. IF

$$\lim_{x\to\infty} f(x) = L, \text{ THEN } \lim_{n\to\infty} a_n = L.$$

THE THEOREM ABOVE TELLS US THAT WE CAN ANALYZE SEQUENCES BY ANALYZING THE "MATCHING" REAL-VALUED FUNCTION.

Ex 5 DETERMINE THE LIMIT OF EACH SEQUENCE.

a) $\left\{5-\frac{1}{n^2}\right\}$ LET $f(x) = 5 - \frac{1}{x^2}$.

SINCE $\lim\limits_{x\to\infty}\left(5-\frac{1}{x^2}\right) = 5$, THE SEQUENCE CONVERGES WITH LIMIT 5.

$$5-\frac{1}{n^2} \longrightarrow 5$$

b) $\left\{\frac{3n^2+5n-8}{6n^2-1}\right\}$ LET $f(x) = \frac{3x^2+5x-8}{6x^2-1}$.

$\lim\limits_{x\to\infty} \frac{3x^2+5x-8}{6x^2-1}$ IS INDETERMINATE.

YOU CAN FIND THE LIMIT BY MULTIPLYING BY $\frac{\frac{1}{x^2}}{\frac{1}{x^2}}$, OR USE L'HÔPITAL'S RULE.

$\lim\limits_{x\to\infty} f(x) = \frac{3}{6}$. SO $a_n \longrightarrow \frac{1}{2}$.

$\boxtimes$

Some important facts above sequence...

1. If a sequence converges, then it must be bounded. That is to say that its terms have both. an upper and a lower bound.

2. A nondecreasing sequence converges if and only if it has an upper bound.

3. A non increasing sequence converges if and only if it has a lower bound.

4. Unbounded sequences cannot converge.

Although we're not going to see examples of these ideas right now, we will use them in the future.

DEFINITION: IF THE TERMS OF ONE SEQUENCE APPEAR IN ANOTHER SEQUENCE IN THEIR GIVEN ORDER, THE FIRST SEQUENCE IS CALLED A <u>SUBSEQUENCE</u> OF THE SECOND.

Ex 6 GIVE SEVERAL SUBSEQUENCE OF THE SEQUENCE $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.

THE ORIGINAL SEQUENCE IS $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$.

HERE ARE A BUNCH OF SUBSEQUENCES...

1) $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\right\} = \left\{\frac{1}{2n}\right\}_{n=1}^{\infty}$

2) $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\right\} = \left\{\frac{1}{n}\right\}_{n=2}^{\infty}$

3) $\left\{\frac{1}{10}, \frac{1}{20}, \frac{1}{30}, \ldots\right\} = \left\{\frac{1}{10n}\right\}_{n=1}^{\infty}$

4) $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \ldots\right\} = \left\{\frac{1}{p}\right\}_{p \text{ PRIME}}$

THEOREM:

1. IF $\{a_n\}$ CONVERGES TO L, THEN ALL SUBSEQUENCES OF $\{a_n\}$ TO L.

2. IF A SUBSEQUENCE OF $\{a_n\}$ DIVERGES, OR IF TWO SUBSEQUENCES CONVERGE TO DIFFERENT LIMITS, THEN $\{a_n\}$ DIVERGES.