

## Section 6.3 - Taylor Series

In Calculus I, we learned how to use a linear function to approximate a more general function. The idea goes by lots of different names: linearization, tangent line approximation, standard linear approximation, etc. For a quick review, see the lecture 20 notes posted at <http://stevekifowit.com/archives/M131/>.

The linearization of  $f$  at  $x = c$  is the function

$$P_1(x) = f(c) + f'(c)(x - c).$$

The approximation  $f(x) \approx P_1(x)$  is the standard linear approximation. It is easy to verify that  $P_1$  is the unique 1st degree polynomial satisfying

$$P_1(c) = f(c), \quad P_1'(c) = f'(c).$$

In other words,  $P_1$  is the only linear function that matches  $f$  and  $f'$  at  $x = c$ .

If you're thinking like a mathematician, you have a natural question: What if we also want  $f''$  to match at  $x = c$ ? Well, then we'd need the quadratic function

$$P_2(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2}(x - c)^2.$$

$P_2$  is the unique polynomial of degree  $\leq 2$  satisfying

$$P_2(c) = f(c), \quad P_2'(c) = f'(c), \quad P_2''(c) = f''(c).$$

The approximation  $f(x) \approx P_2(x)$  is the standard quadratic approximation.

We can continue indefinitely (as long as  $f$  has enough derivatives). The polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is the unique polynomial of degree  $\leq n$  satisfying

$$P_n(c) = f(c), \quad P_n'(c) = f'(c), \quad P_n''(c) = f''(c), \quad \dots, \quad P_n^{(n)}(c) = f^{(n)}(c).$$

### Definition 1

Suppose  $f$  has  $n$  derivatives at  $x = c$ , then the  $n$ th Taylor polynomial for  $f$  at  $x = c$  is the polynomial (of degree  $\leq n$ ) given by

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \frac{f'''(c)}{3!}(x - c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n.$$

If  $c = 0$ , the Taylor polynomial is called a Maclaurin polynomial. Taylor and Maclaurin polynomials approximate functions in the sense described above.

### Example 1

Find the 4th Maclaurin polynomial for  $f(x) = e^x$ .

The Maclaurin polynomial is the Taylor polynomial with  $c = 0$ .

$$\begin{aligned}
f(x) &= e^x, & f(0) &= 1 \\
f'(x) &= e^x, & f'(0) &= 1 \\
f''(x) &= e^x, & f''(0) &= 1 \\
f'''(x) &= e^x, & f'''(0) &= 1 \\
f^{(4)}(x) &= e^x, & f^{(4)}(0) &= 1
\end{aligned}$$

It follows that

$$P_4(x) = 1 + (x - 0) + \frac{1}{2!}(x - 0)^2 + \frac{1}{3!}(x - 0)^3 + \frac{1}{4!}(x - 0)^4$$

or

$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4. \quad \diamond$$

## Example 2

Find the 4th Taylor polynomial for  $f(x) = \ln x$  at  $x = 1$ . Sketch the graph of  $f$  and  $P_4$ .

$$\begin{aligned}
f(x) &= \ln x, & f(1) &= 0 \\
f'(x) &= \frac{1}{x}, & f'(1) &= 1 \\
f''(x) &= -\frac{1}{x^2}, & f''(1) &= -1 \\
f'''(x) &= \frac{2}{x^3}, & f'''(1) &= 2 \\
f^{(4)}(x) &= -\frac{6}{x^4}, & f^{(4)}(1) &= -6
\end{aligned}$$

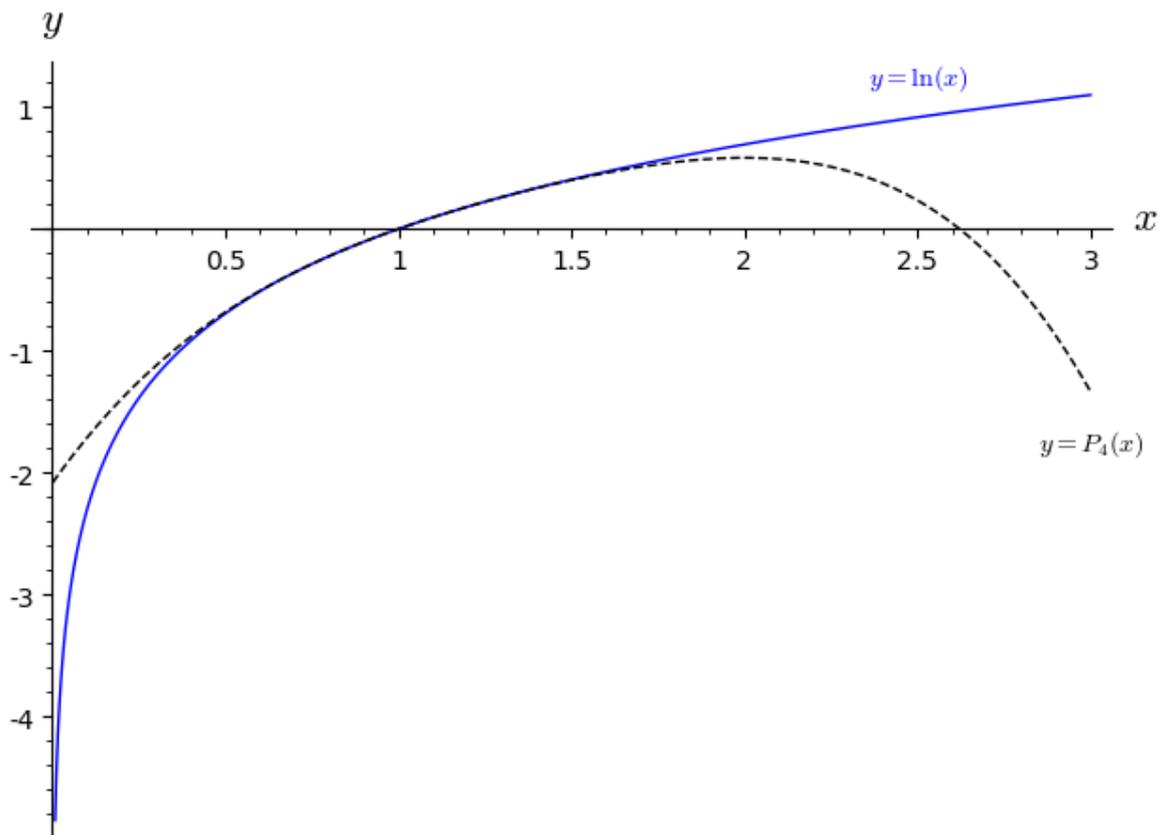
It follows that

$$P_4(x) = 0 + 1(x - 1) - \frac{1}{2!}(x - 1)^2 + \frac{2}{3!}(x - 1)^3 - \frac{6}{4!}(x - 1)^4$$

or

$$P_4(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4.$$

The graphs of  $y = \ln(x)$  and  $y = P_4(x)$  are shown below.  $\diamond$



## Definition 2

Suppose  $f$  has derivatives of all orders at  $x = c$ , then the Taylor series for  $f$  at  $x = c$  is the infinite series given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!} (x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!} (x - c)^n + \cdots$$

If  $c = 0$ , the Taylor series is called a Maclaurin series.

## Example 3

Find the Maclaurin series for  $f(x) = e^x$ .

Let's look back at example 1 and generalize from the pattern:

$$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \cdots + \frac{1}{n!}x^n + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \quad \diamond$$

## Example 4

Find the Taylor series for  $f(x) = \ln x$  at  $x = 1$ .

Once again, let's look at example 2 and generalize from the pattern. We will find that the series is

$$(x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x - 1)^n.$$

Now take a minute to look back at example 5 in the lecture notes for section 6.1. In that example, if you replace every  $x$  by  $x - 1$ , you'll obtain precisely the Taylor series above. That means that the power series centered at  $x = 1$  is the same as the Taylor series at  $x = 1$ . This is not a coincidence.  $\diamond$

## Theorem 1

Suppose  $f$  is represented by a power series centered at  $x = c$ . Also suppose that the power series converges to  $f$  on an open interval containing  $c$ . Then the power series centered at  $x = c$  is identical to the Taylor at  $x = c$ .

## Comments

1. Based on the theorem, we can find the power series for a function by finding the Taylor series for the function. That is awesome! We now have a straight-forward procedure for finding the power series representation for function.
2. The power series representation for a function (which can be obtained from the Taylor series) does not necessarily converge to the function. This is a very important point, and it may sound rather odd. So far we have not seen anything that guarantees that a power series for a function actually converges to that function! We will come back to that later.

## Example 5

Find a power series representation centered at  $x = 0$  for  $f(x) = \sin x$ .

We find the Maclaurin series for  $f(x) = \sin x$ ...

$$\begin{aligned} f(x) &= \sin x, & f(0) &= 0 \\ f'(x) &= \cos x, & f'(0) &= 1 \\ f''(x) &= -\sin x, & f''(0) &= 0 \\ f'''(x) &= -\cos x, & f'''(0) &= -1 \\ & & \vdots & \end{aligned}$$

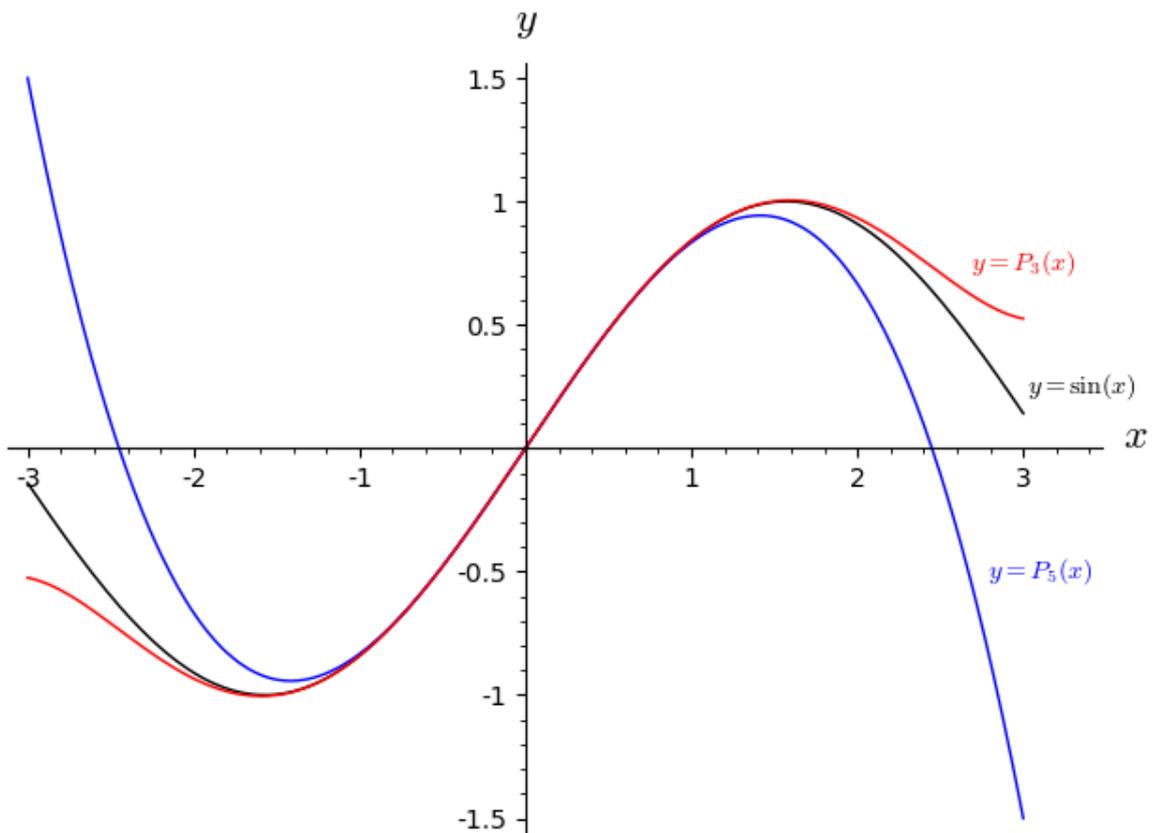
Continuing, we will find that all even-order derivatives at  $x = 0$  are zero, and the odd-order derivatives at  $x = 0$  alternate from 1 to  $-1$ . So our Maclaurin series is

$$0 + 1(x - 0) + \frac{0}{2!}(x - 0)^2 - \frac{1}{3!}(x - 0)^3 + \frac{0}{4!}(x - 0)^4 + \frac{1}{5!}(x - 0)^5 + \frac{0}{6!}(x - 0)^6 - \frac{1}{7!}(x - 0)^7 + \dots,$$

and upon cleaning this up, we get

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}. \quad \diamond$$

Shown below are the graphs of  $y = \sin x$  and some of the partial sums of its Maclaurin series.



## Theorem 2 (Taylor's theorem)

Suppose  $f$  is differentiable through order  $n + 1$  on an interval  $I$  containing  $c$ . Let  $P_n(x)$  be the  $n$ th Taylor polynomial for  $f$  at  $x = c$ . Then for each  $x$  in  $I$ , there exists a number  $z$  between  $x$  and  $c$  such that

$$f(x) = P_n(x) + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} (x-c)^{n+1}.$$

$R_n(x)$  is called the  $n$ th remainder.

## Example 6

Suppose you use the 4th Maclaurin polynomial for  $e^x$  to approximate  $e^{0.1}$ . Use Taylor's theorem to find an upper bound on the error in your approximation.

For  $f(x) = e^x$ , the 4th Maclaurin polynomial is  $P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$  (see example 1).

According to Taylor's theorem (with  $c = 0$  and  $n = 4$ ), there is a number  $z$  between 0 and 0.1 such that

$$e^{0.1} = P_4(0.1) + \frac{f^{(5)}(z)}{120} (0.1)^5.$$

Now,  $f^{(5)}(x) = e^x$  so that  $f^{(5)}(z) = e^z$ . Furthermore, since  $e^x$  is an increasing function, it follows that

$$e^{0.1} - P_4(0.1) = \frac{e^z}{120}(0.1)^5 \leq \frac{e^{0.1}}{120}(0.1)^5.$$

The upper bound on the right *could be* computed by using a calculator, but stop and think about that! This problem is all about approximating  $e^{0.1}$ . If we use our calculator to compute it, it defeats the purpose of the problem. Do you understand?

Rather than compute  $e^{0.1}$  in order to arrive at a bound, let's just use some information we already know:

$$e^{0.1} < e \approx 2.71828 < 3.$$

Therefore the upper bound satisfies

$$\frac{e^{0.1}}{120}(0.1)^5 < \frac{3}{120}(0.1)^5 = 2.5 \times 10^{-7}.$$

That means the difference between  $e^{0.1}$  and  $P_4(0.1) = 1.1051708\bar{3}$  is smaller than 0.00000025. That's a pretty good approximation!  $\diamond$

## Example 7

Suppose you use the 3rd Maclaurin polynomial for  $\sin x$  to approximate  $\sin(0.125)$ . Use Taylor's theorem to find an upper bound on the error in your approximation.

Using Taylor's theorem and looking back to example 5, we have

$$\sin(0.125) = P_3(0.125) + \frac{-\cos z}{4!}(0.125)^4,$$

for some number  $z$  between 0 and 0.125.

Since  $-1 \leq \cos z \leq 1$ , it follows that

$$|\sin(0.125) - P_3(0.125)| \leq \frac{1}{4!}(0.125)^4 \approx 1.017252 \times 10^{-5}.$$

So the difference between  $\sin(0.125)$  and  $P_3(0.125) = (0.125) - (0.125)^3/6 \approx 0.1246744$  is less than 0.000011.  $\diamond$

## Theorem 3 (Convergence of Taylor series)

Suppose  $f$  has derivatives of all orders on an interval  $I$  containing  $c$ . Then the Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

converges to  $f(x)$  for all  $x$  in  $I$  if and only if

$$\lim_{n \rightarrow \infty} R_n(x) = 0$$

for all  $x$  in  $I$ .

## Example 8

Find the Maclaurin series for  $f(x) = \cos x$ . Then prove that the Maclaurin series converges to the function at any value of  $x$ .

We find the Maclaurin series for  $f(x) = \cos x...$

$$\begin{aligned}
f(x) &= \cos x, & f(0) &= 1 \\
f'(x) &= -\sin x, & f'(0) &= 0 \\
f''(x) &= -\cos x, & f''(0) &= -1 \\
f'''(x) &= \sin x, & f'''(0) &= 0 \\
& \vdots
\end{aligned}$$

Continuing, we will find that all odd-order derivatives at  $x = 0$  are zero, and the even-order derivatives at  $x = 0$  alternate from 1 to  $-1$ . So our Maclaurin series is

$$1 + 0(x - 0) - \frac{1}{2!}(x - 0)^2 + \frac{0}{3!}(x - 0)^3 + \frac{1}{4!}(x - 0)^4 + \frac{0}{5!}(x - 0)^5 - \frac{1}{6!}(x - 0)^6 + \frac{0}{7!}(x - 0)^7 + \dots,$$

and upon cleaning this up, we get

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

Next, in order to prove convergence to  $f(x) = \cos x$ , we must show that the  $n$ th remainder approaches zero as  $n$  approaches infinity.

$R_n(x)$  has the form

$$R_n(x) = \frac{f^{(n)}(z)}{(n+1)!} x^{n+1},$$

and since the  $n$ th derivative is a sine or a cosine, its value is between  $-1$  and  $1$ . Therefore

$$|R_n(x)| = \frac{|f^{(n)}(z)|}{(n+1)!} |x|^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!}.$$

It is probably not obvious that the expression on the right approaches zero as  $n \rightarrow \infty$ . Here is a clever way to see it.

Look back at example 4 in the lecture notes for section 6.1. In that example, we used the ratio test to show that the power series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges absolutely for all  $x$ . By the  $n$ th term test for divergence, it follows that the terms of the series approach zero. So we must have

$$\lim_{n \rightarrow \infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

Our final conclusion is that for any number  $x$ ,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \quad \diamond$$

## Example 9

Functions do not necessarily converge to their series representations. Here is a classic example due to Cauchy.

Consider the function

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Although we will not work through the details here,  $f$  is infinitely differentiable, and it has a Maclaurin series that converges for all real numbers. In fact, its Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} 0 x^n = 0.$$

Its Maclaurin series is identically zero! On the other hand,  $f(x)$  is nonzero everywhere except at  $x = 0$ . Even though both are defined everywhere,  $f(x)$  is equal to its Maclaurin series only at one point.  $\diamond$