Math 131 - Test 3

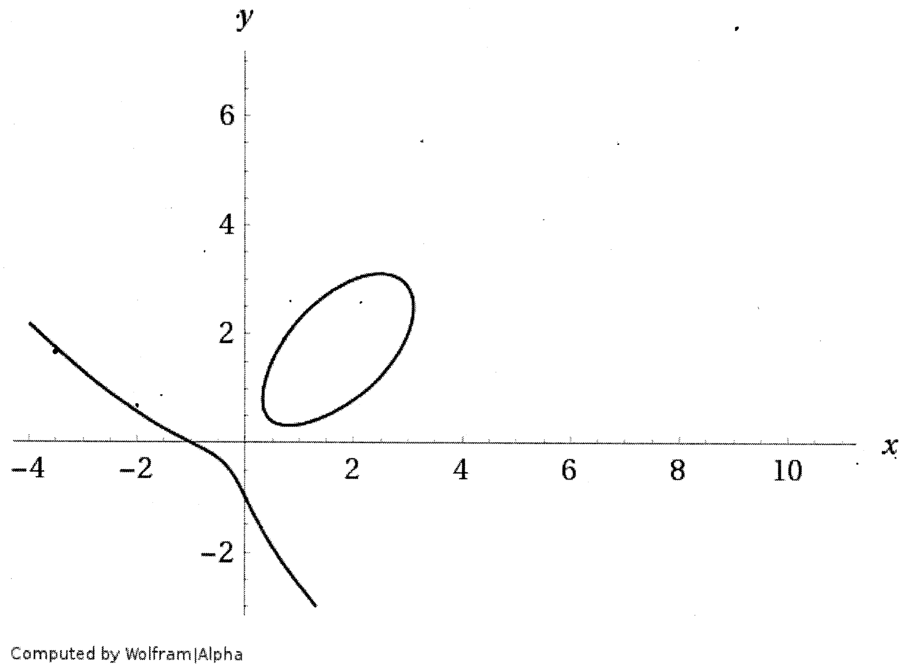
November 18, 2020

Name Key

Score ________

Show all work to receive full credit. Supply explanations where necessary. This test is due November 30 by email. **You must work individually on this test.**

1. (10 points) The graph of the equation $x^3 + y^3 = 6xy - 1$ is shown below. Find an equation of the normal line at the point $(2, 3)$.



$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(6xy-1)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6x\frac{dy}{dx} + 6y$$

$$(3y^2 - 6x)\frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x}$$

$$\frac{dy}{dx} = \frac{2y-x^2}{y^2-2x}$$

$$m = \left.\frac{dy}{dx}\right|_{(2,3)} = \frac{2}{5}$$

$$m_\perp = -\frac{5}{2}$$

$$y - 3 = -\frac{5}{2}(x-2)$$

or

$$y = -\frac{5}{2}x + 8$$

2. (5 points) Let $f(x) = x^5 + 2x^3 + x$. Find $(f^{-1})'(-4)$.

$f'(x) = 5x^4 + 6x^2 + 1$

$$(f^{-1})'(-4) = \frac{1}{f'(f^{-1}(-4))}$$

$f^{-1}(-4) = z \Leftrightarrow f(z) = -4$

$z = -1$ BY OBSERVATION.

$$= \frac{1}{f'(-1)} = \boxed{\frac{1}{12}}$$

3. (6 points) Find the slope of the line tangent to the graph of $y = (\cos^{-1} x)^2$ at the point where $x = 1/\sqrt{2}$.

$$\frac{dy}{dx} = 2(\cos^{-1}x)\frac{d}{dx}\cos^{-1}x = 2\cos^{-1}x\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{-2\cos^{-1}x}{\sqrt{1-x^2}}$$

$$m = \left.\frac{dy}{dx}\right|_{x=\frac{1}{\sqrt{2}}} = \frac{-2\left(\frac{\pi}{4}\right)}{\sqrt{1/2}} = \boxed{-\frac{\pi}{\sqrt{2}}}$$

4. (8 points) Use logarithmic differentiation to find dy/dx: $\quad y = \dfrac{\sqrt{(2x+1)(3x+2)}}{(4x+3)^2}$

$$\ln y = \frac{1}{2}\ln(2x+1) + \frac{1}{2}\ln(3x+2) - 2\ln(4x+3)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left(\frac{2}{2x+1}\right) + \frac{1}{2}\left(\frac{3}{3x+2}\right) - 2\left(\frac{4}{4x+3}\right)$$

$$\boxed{\frac{dy}{dx} = \left[\frac{1}{2x+1} + \frac{3/2}{3x+2} - \frac{8}{4x+3}\right]\frac{\sqrt{(2x+1)(3x+2)}}{(4x+3)^2}}$$

5. (4 points) Find $g'(x)$ if $g(x) = 2^{\cot x}$.

$$g'(x) = 2^{\cot x}(\ln 2)\frac{d}{dx}(\cot x)$$

$$= 2^{\cot x}(\ln 2)(-\csc^2 x)$$

$$= \boxed{-(\ln 2)(\csc^2 x)\, 2^{\cot x}}$$

6. (7 points) Find the linearization of $h(x) = \sqrt[3]{x} + \sqrt[5]{x} + \sqrt[7]{x}$ at $x = 1$. Then use your linearization to approximate $h(1.1)$.

$$L(x) = h(1) + h'(1)(x-1)$$

$$h(1) = 3; \quad h'(x) = \frac{1}{3}x^{-2/3} + \frac{1}{5}x^{-4/5} + \frac{1}{7}x^{-6/7}$$

$$h'(1) = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} = \frac{71}{105}$$

$$\boxed{L(x) = 3 + \frac{71}{105}(x-1)}$$

$$h(1.1) \approx L(1.1) = 3 + \frac{7.1}{105}$$

$$\approx 3.067619$$

7. (7 points) Find all critical numbers of $f(x) = 5x^{3/7} - 2x^{10/7}$. (Helpful hint: Simplify your derivative by factoring out $x^{-4/7}$.)

$$f'(x) = \frac{15}{7}x^{-4/7} - \frac{20}{7}x^{3/7} = \frac{1}{7}x^{-4/7}(15 - 20x)$$

$$= \frac{15 - 20x}{7x^{4/7}}$$

$f'(x)$ DNE WHEN $x = 0$

$f'(x) = 0$ WHEN $15 - 20x = 0$

$$x = \frac{15}{20} = \frac{3}{4}$$

CRIT #'s ARE

$x = 0$, $x = \frac{3}{4}$

8. (8 points) Use calculus techniques to find the absolute extreme values of $f(x) = 1 - 2\sin x$ on the interval $[-1, 2]$.

$f'(x) = -2\cos x$

$f'(x) = 0 \Rightarrow \cos x = 0$

$\Rightarrow x = \frac{\pi}{2}$

(ONLY SOLUTION ON $[-1,2]$)

CRIT #'S AND ENDPTS ARE $x = -1, 2, \frac{\pi}{2}$

x	$f(x)$	
-1	$1 + 2\sin(1) \approx 2.68$	← ABS MAX.
2	$1 - 2\sin(2) \approx -0.82$	
$\frac{\pi}{2}$	-1	← ABS MIN.

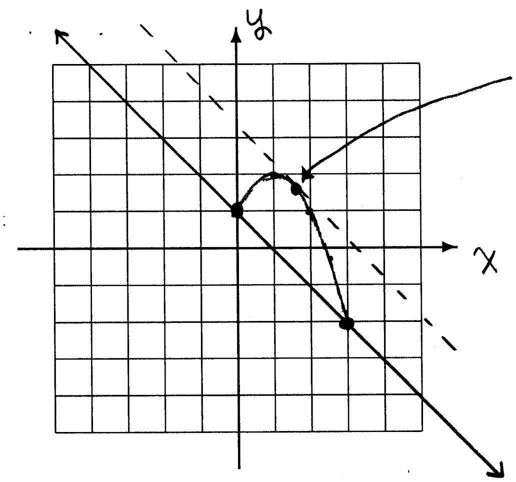
9. (6 points) Let $f(x) = 2 - (x-1)^2$ on $[0, 3]$. Carefully sketch the graph of f on $[0, 3]$. Then find a point at which the instantaneous rate of change of f is equal to the average rate of change of f. (Helpful hint: Refer to the Mean Value Theorem.)

$f'(x) = -2(x-1)$

$-2(x-1) = -1$

$\Rightarrow x = \frac{3}{2}$

$\boxed{x = \frac{3}{2}}$



$x = \frac{3}{2}$

SLOPE = AVG RATE OF CHANGE

$\frac{f(3) - f(0)}{3} = \frac{-3}{3} = -1$

10. (6 points) Let $g(x) = x^4 + \cos(20x)$. Without looking at the graph of g, determine whether the graph is concave up or concave down at the point where $x = 0.7$.

$g'(x) = 4x^3 - 20\sin 20x$

$g''(x) = 12x^2 - 400\cos 20x$

$g''(0.7) \approx -48.815 < 0 \Rightarrow$ GRAPH IS CD.

11. (10 points) Find open intervals on which the graph of $f(x) = 2x^4 - 16x^2 + 3$ is increasing/decreasing. Also identify all relative extreme values.

$$f'(x) = 8x^3 - 32x = 8x(x^2-4)$$
$$= 8x(x-2)(x+2)$$

$f'(x)$ DNE NOWHERE.

$f'(x) = 0 \Rightarrow x = 0, 2, -2$

SIGNS OF f'	$-$	$+$	$-$	$+$
		-2	0	2
f IS	DEC	INC	DEC	INC

f IS INCREASING ON $(-2,0) \cup (2,\infty)$.

f IS DECREASING ON $(-\infty,-2) \cup (0,2)$

$f(-2) = -29$ IS A REL MIN

$f(0) = 3$ IS A REL MAX

$f(2) = -29$ IS A REL MIN

12. (12 points) Let $f(x) = (x-6)^3(x-2)$. Find $f''(x)$ and write it in factored form. Then find open intervals on which the graph of f is concave up/down. Identify all points of inflection of the graph of f.

$$f'(x) = 3(x-6)^2(x-2) + (x-6)^3 \quad \text{(From product rule)}$$
$$= (x-6)^2[3x-6+x-6] = (x-6)^2(4x-12)$$
$$f''(x) = 2(x-6)(4x-12) + (x-6)^2(4) \quad \text{(From product rule)}$$
$$= 4(x-6)[2x-6+x-6] = 4(x-6)(3x-12)$$

$f''(x) = 0 \Rightarrow x = 6,\ x = 4$

SIGNS OF f''	$+$	$-$	$+$
		4	6
GRAPH IS	CU	CD	CU

GRAPH IS CD ON $(4,6)$ AND CU ON $(-\infty,4) \cup (6,\infty)$.

$(4,-16)$ AND $(6,0)$ ARE INFLECTION PTS.

13. (5 points) Evaluate the limit: $\lim_{x\to-\infty} \dfrac{7x-13}{\sqrt{4x^2-3x-8}}$

$\sqrt{x^2} = |x| = -x$ FOR $x<0$

$$\lim_{x\to-\infty} \frac{7x-13}{\sqrt{4x^2-3x-8}} \cdot \frac{\frac{1}{-x}}{\frac{1}{\sqrt{x^2}}} = \lim_{x\to-\infty} \frac{-7+\frac{13}{x}}{\sqrt{4-\frac{3}{x}-\frac{8}{x^2}}}$$

$$= \frac{-7}{\sqrt{4}} = -\frac{7}{2}$$

14. (4 points) Find all horizontal and vertical asymptotes of the graph of $H(x) = \dfrac{3x^2+7x}{(x-3)(2x+1)}$.

H.A: DEGREE OF NUMER = DEGREE OF DENOM

$\Rightarrow$ H.A. FOLLOWS FROM RATIO OF LEADING TERMS.

$\Rightarrow$ $\boxed{\text{H.A. IS } y = \frac{3}{2}}$

V.A: $H(x) = \dfrac{x(3x+7)}{(x-3)(2x+1)}$

$x=3$ AND $x=-\frac{1}{2}$ ARE ZEROS OF DENOM BUT NOT ZEROS OF NUMER $\Rightarrow$

V.A.: $x=3$, $x=-\frac{1}{2}$

15. (2 points) Evaluate the limit: $\lim_{x\to\infty} \dfrac{\cos x}{x}$

$-1 \le \cos x \le 1 \Rightarrow -\frac{1}{x} \le \frac{\cos x}{x} \le \frac{1}{x}$

FOR $x>0$

LET $x \to \infty$ TO GET

$$0 \le \lim_{x\to\infty} \frac{\cos x}{x} \le 0$$

OR $\boxed{\lim_{x\to\infty} \frac{\cos x}{x} = 0}$