Math 131 - Final Exam

December 16, 2020

Name key

Score

Show all work to receive full credit. Carefully read the directions on the preceding page. This test is due no later than December 17 at 11:59 pm.

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1. (5 points) Determine the limit. Show analytically (not with a graph or table) how you got your answer.

$$\lim_{x\to 2}\left(\frac{3x+5}{\underbrace{x^3-4x^2+4x}_{x(x-2)^2}}\right)$$

$\frac{11}{0}$ FORM. CHECK SIDED LIMITS FOR $\pm\infty$.

$$\lim_{x\to 2^+}\frac{3x+5}{x(x-2)^2} = +\infty$$

TO THE RIGHT OF $x=2$, THIS IS POSITIVE.

$$\lim_{x\to 2^-}\frac{3x+5}{x(x-2)^2} = +\infty$$

TO THE LEFT OF $x=2$, THIS IS POSITIVE.

$\boxed{\text{LIMIT IS } +\infty.}$

2. (5 points) Determine the limit. Use algebraic techniques (not a graph, table, or L'Hôpital's rule) to show how you got your answer.

$$\lim_{x\to 1}\left(\frac{7x-7}{1-\sqrt{x}}\right)$$

0/0 FORM. MORE WORK.

$$\lim_{x\to 1}\frac{7(x-1)}{1-\sqrt{x}}\cdot\frac{1+\sqrt{x}}{1+\sqrt{x}} = \lim_{x\to 1}\frac{7(x-1)(1+\sqrt{x})}{1-x} = \lim_{x\to 1} -7(1+\sqrt{x})$$

$$= -14$$

$\boxed{\text{LIMIT IS } -14.}$

3. (5 points) Yes or No: Is f continuous at $x = 5$? Use the definition of continuity to support your answer.

$$f(x) = \begin{cases} x^2 - x - 8, & x < 5 \\ 12, & x = 5 \\ \dfrac{3(x-20)+9x}{x-5}, & x > 5 \end{cases}$$

$$\lim_{x \to 5^-} f(x) = (5)^2 - 5 - 8 = 25 - 5 - 8 = 12$$

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} \left(\frac{12x - 60}{x-5} \right) = \lim_{x \to 5^+} \frac{12(x-5)}{x-5} = 12$$

$$f(5) = 12$$

$$\boxed{\text{Yes. } \lim_{x \to 5} f(x) = f(5)}$$

4. (5 points) Let $f(x) = x^2 - 5x$. Write $f'(x)$ in the box, then use the limit definition of derivative to obtain your answer.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^2 - 5(x+h)] - [x^2 - 5x]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{5x} - 5h - \cancel{x^2} + \cancel{5x}}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x + h - 5)}{\cancel{h}}$$

$$= 2x - 5$$

$$\boxed{f'(x) = 2x - 5}$$

5. (5 points) Refer to the table below. Let $h(x) = \dfrac{3x + p(x)}{q(x)}$ and compute $h'(2)$.

$$h'(x) = \frac{q(x)\left[3+p'(x)\right] - \left[3x+p(x)\right]q'(x)}{\left[q(x)\right]^2}$$

x	0	1	2
$p(x)$	3	1	5
$p'(x)$	-2	-1	-4
$q(x)$	0	4	3
$q'(x)$	8	0	-2

$$h'(2) = \frac{(3)(3+(-4)) - (6+5)(-2)}{(3)^2}$$

$$= \frac{-3+22}{9} = \frac{19}{9}$$

$$\boxed{h'(2) = \frac{19}{9}}$$

6. (5 points) Refer to the function p described in the table above (see the preceding problem). Let $f(x) = e^{x\,p(x)}$, and compute $f'(1)$.

$$f'(x) = e^{x\,p(x)}\left(x\,p'(x) + p(x)\right)$$

$$f'(1) = e^{p(1)}\left(p'(1) + p(1)\right) = e(-1+1) = e(0) = 0$$

$$\boxed{f'(1) = 0}$$

7. (5 points) An object is launched straight upward from the ground with an initial velocity of 47.3 meters per second. What is the maximum height of the object? (Ignore all forces except gravity, and use $g = 9.8$ m/sec^2.)

$$s(t) = -4.9t^2 + 47.3t + 0$$

$$s'(t) = -9.8t + 47.3 = 0 \Rightarrow t = \frac{47.3}{9.8}$$

$$\boxed{\text{MAX HEIGHT} = s\left(\frac{47.3}{9.8}\right) \approx 114.1 \text{ m}}$$

8. (5 points) Find an equation of the line tangent to the graph of $xy^2 = x^2 - 2y$ at the point $(2, 1)$.

$$\frac{d}{dx}(xy^2) = \frac{d}{dx}(x^2 - 2y)$$

$$y^2 + 2xy\frac{dy}{dx} = 2x - 2\frac{dy}{dx}$$

$$(2xy+2)\frac{dy}{dx} = 2x - y^2$$

$$\frac{dy}{dx} = \frac{2x-y^2}{2xy+2}, \quad m = \frac{2(2)-1}{2(2)(1)+2} = \frac{3}{6} = \frac{1}{2}$$

$$\boxed{\text{TAN. LINE: } y - 1 = \tfrac{1}{2}(x-2) \quad \text{OR} \quad y = \tfrac{1}{2}x}$$

9. (5 points) Let $g(x) = x^{3x}$. Use logarithmic differentiation to compute $g'(2)$. Round your final answer to the nearest hundredth.

$$y = x^{3x}$$

$$\ln y = 3x \cdot \ln x$$

$$\frac{1}{y}y' = 3\ln x + \frac{3x}{x}$$

$$y' = (3\ln x + 3)y$$

$$g'(x) = (3\ln x + 3)x^{3x}$$

$$g'(2) = (3\ln 2 + 3)(2^6)$$

$$\boxed{g'(2) \approx 325.08}$$

10. (5 points) The graph of f passes through the point $(3, 5)$, and the tangent line at that point has slope -7. Find the linearization of f at $x = 3$ and use it to approximate $f(2.9)$.

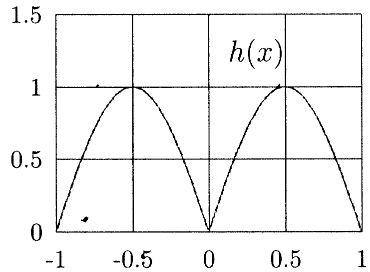
$$L(x) = f(3) + f'(3)(x-3)$$

$$L(x) = 5 + (-7)(x-3) = -7x + 26$$

$$\boxed{L(x) = -7x + 26, \quad f(2.9) \approx L(2.9) = 5.7}$$

11. (5 points) The function f is defined on the interval $[-1, 1]$. The graph of f is shown below. Use the graph to find the critical points of f. Say why each is a critical point.

CRIT POINTS ARE DOMAIN INTERIOR PTS WHERE $f'(x) = 0$ OR $f'(x)$ DNE.



IN $(-1,1)$,

$f'(x) = 0$ AT $X = \pm 0.5$

$f'(x)$ DNE AT $X = 0$

SHARP POINT.

$\boxed{\text{3 CRIT. POINTS IN } (-1,1) \ldots \quad f'(x) = 0 \text{ AT } X = \pm 0.5, \ f'(x) \text{ DNE AT } X = 0}$

12. (5 points) Let $f(x) = \frac{1}{2}x - x^{2/3}$ on $[-1, 4]$. Use calculus to find the absolute extreme values of f.

$$f'(x) = \frac{1}{2} - \frac{2}{3}X^{-1/3}$$

$$f'(x) = 0 \Rightarrow \frac{2}{3}X^{-1/3} = \frac{1}{2}$$

$$X^{-1/3} = \frac{3}{4}$$

$$X = \left(4/3\right)^3 \approx 2.37$$

$f'(x)$ DNE WHEN $X = 0$

X	f(x)
0	0
$(4/3)^3$	-0.5926

X	f(x)
-1	-1.5
4	-0.5198

$\boxed{\text{ABS MAX: } f(0) = 0 \quad \text{ABS MIN: } f(-1) = -1.5}$

13. (5 points) Evaluate the limit: $\lim_{x\to\infty} xe^{-x}$ $\quad \infty \cdot 0$ FORM

$$\underbrace{\lim_{X\to\infty} \frac{X}{e^X} = \lim_{X\to\infty} \frac{1}{e^X}}_{\text{L'HÔPITAL'S}} = 0$$

$\boxed{\text{LIMIT IS } 0.}$

14. (5 points) The functions $f(x)$ and $f'(x)$ are defined for all x. Furthermore, $f'(x)$ has exactly three zeros: $x = -3$, $x = 5$, and $x = 7$. Use the information below to find the locations (x-values) of all relative extrema.

x	-12	-6	0	6	12
$f'(x)$	3	8	2	1	-5

1ST DERIVATIVE TEST...

SIGNS OF f' FROM TABLE: $+$ (left of -3), $+$ (between -3 and 5), $+$ (between 5 and 7), $-$ (right of 7)

-3 $\quad 5 \quad 7$

f IS: INC, INC, INC, DEC

ONLY ONE RELATIVE EXTREME VALUE: AT $X=7$

$$\boxed{f(7) \text{ IS A RELATIVE MAX.}}$$

15. (5 points) Find the inflection points of the graph of $g(x) = 3x^5 + 5x^4 + 2x - 25$.

$g'(x) = 15x^4 + 20x^3 + 2$

$g''(x) = 60x^3 + 60x^2$

$= 60x^2(x+1)$

$g''(x) = 0 \Rightarrow X = 0$

$X = -1$

SIGNS OF g'': $-$ (left of -1), $+$ (between -1 and 0), $+$ (right of 0)

$-1 \quad 0$

GRAPH IS: CD, CU, CU

INF POINT AT $X = -1$

(ONLY ONE!)

$$\boxed{(-1, g(-1)) = (-1, -25)}$$

16. (5 points) Find $f(x)$ if $f'(x) = \dfrac{2}{\sqrt{1-x^2}}$ and $f(1/2) = \pi$.

$$f(x) = \int \frac{2}{\sqrt{1-x^2}}\,dx = 2\sin^{-1}x + C \qquad f\left(\tfrac{1}{2}\right) = 2\sin^{-1}\left(\tfrac{1}{2}\right) + C$$

$$= 2\left(\tfrac{\pi}{6}\right) + C = \tfrac{\pi}{3} + C$$

$$= \pi$$

$$\Downarrow$$

$$\boxed{f(x) = 2\sin^{-1}x + \frac{2\pi}{3}.} \qquad C = \frac{2\pi}{3}$$

17. (5 points) Some values of the function f are given below. Use 4 subintervals of equal length and left endpoints of the subintervals to compute a Riemann sum for f on $[0, 1]$.

x	0.00	0.15	0.25	0.50	0.75	0.90	1.00
$f(x)$	1.00	0.70	0.52	0.13	−0.08	−0.07	0.00

$\Delta x = 0.25$

PARTITION IS

$0 < 0.25 < 0.5 < 0.75 < 1$

$\uparrow c_1 \quad \uparrow c_2 \quad \uparrow c_3 \quad \uparrow c_4$

C_k'S ARE LEFT ENDPTS.

RIEMANN SUM IS $\displaystyle\sum_{k=1}^{4} f(c_k)\,(0.25)$

$$= (0.25)\big(1.00 + 0.52 + 0.13 + (-0.08)\big)$$

$$= 0.3925$$

$$\boxed{0.3925}$$

18. (5 points) Evaluate the definite integral. Write your answer in exact form.

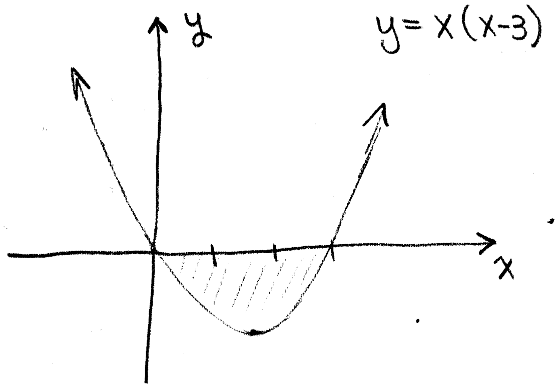
$$\int_1^4 \left(\frac{1}{x} + e^x + \sqrt{x}\right) dx = \int_1^4 \left(\frac{1}{x} + e^x + x^{1/2}\right) dx$$

$$= \left(\ln x + e^x + \frac{2}{3}x^{3/2}\right)\Bigg|_1^4 =$$

$$= \left(\ln 4 + e^4 + \frac{16}{3}\right) - \left(0 + e + \frac{2}{3}\right)$$

$$\boxed{\ln 4 + e^4 - e + \frac{14}{3} \approx 57.93}$$

19. (5 points) Use a definite integral to find the area of the bounded region between the graph of $y = x^2 - 3x$ and the x-axis. Write your answer in exact form.



$$\text{AREA} = -\int_0^3 (x^2 - 3x)\,dx$$

$$= -\left[\frac{1}{3}x^3 - \frac{3}{2}x^2\right]_0^3$$

$$= -\left(\left[9 - \frac{27}{2}\right] - 0\right) = \frac{9}{2}$$

$$\boxed{\text{AREA} = 9/2}$$

20. (5 points) In order to evaluate the following integral, an appropriate u-substitution should be made. Carry out the substitution and write the new integral. Do not evaluate the new integral.

$$\int \frac{\log_7 \sqrt{x}}{\sqrt{x}}\,dx$$

$$u = \sqrt{x} = x^{1/2}$$

$$du = \frac{1}{2}x^{-1/2}\,dx$$

$$2\,du = \frac{1}{\sqrt{x}}\,dx$$

$$\boxed{\int \frac{\log_7 \sqrt{x}}{\sqrt{x}}\,dx = \int 2 \log_7 u\,du}$$