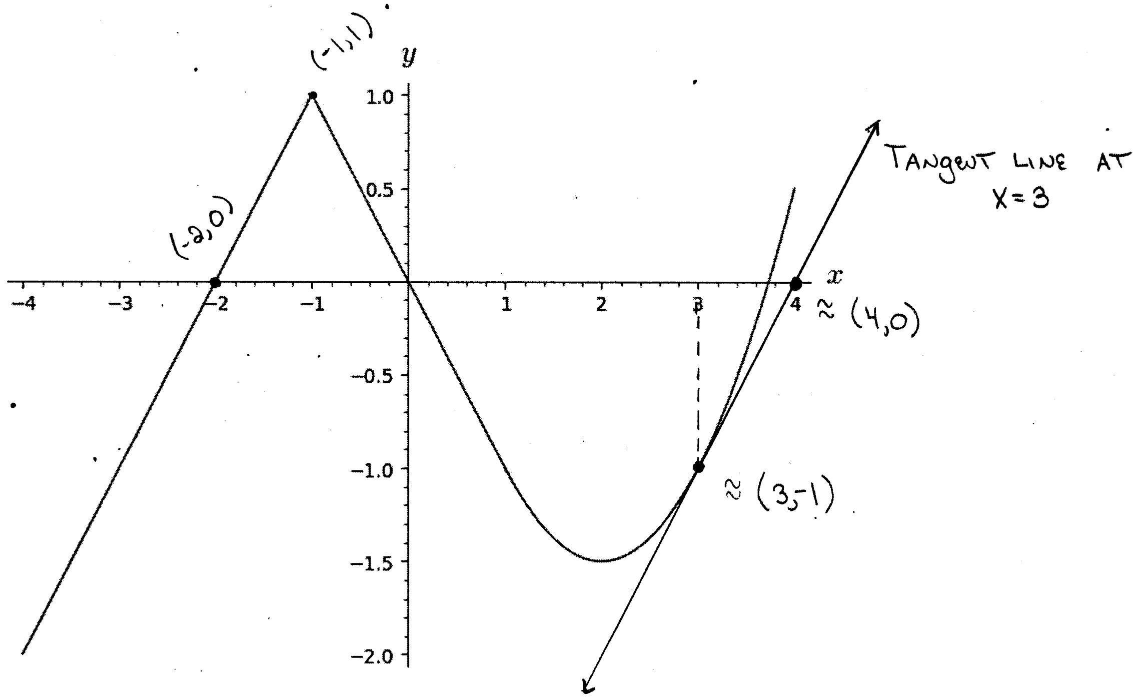
Math 131 - Test 2
October 13, 2021

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary. Unless otherwise indicated, use differentiation rules for all derivatives, and do not simplify.

1. (12 points) The graph of $y = f(x)$ is shown below. Use the graph for each part of this problem.



(a) Sketch the tangent line at $x = 3$. Then use your tangent line to estimate $f'(3)$. Show work or explain your reasoning.

SLOPE OF TAN. LINE $\approx \dfrac{0-(-1)}{4-3} = \boxed{1}$ (SEE PTS. ABOVE)

(b) Identify a point at which $f'(x) = 0$. Explain how you know.

$f'(x) = 0$ WHEN $\boxed{x=2}$. TANGENT LINE IS HORIZONTAL AT THAT POINT.

(c) Identify a point at which $f'(x)$ does not exist. Explain how you know.

$f'(x)$ DNE AT $\boxed{x=-1}$. THE SHARP POINT INDICATES THAT THE SLOPE FROM LEFT $\neq$ SLOPE FROM RIGHT.

(d) Determine the value of $f'(-2)$. Show work or explain your reasoning.

$f'(-2) = \dfrac{1-0}{-1-(-2)} = \boxed{1}$ THE GRAPH AROUND $x=-2$ IS A LINE WITH SLOPE 1.

2. (10 points) Let $f(x) = 2x^2 + 3x$. Use a limit definition of the derivative to compute $f'(2)$. Show all work.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[2(x+h)^2 + 3(x+h)] - [2x^2 + 3x]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 + \cancel{3x} + 3h - \cancel{2x^2} - \cancel{3x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4x + 2h + 3)}{\cancel{h}}$$

$$= \lim_{h \to 0} (4x + 2h + 3) = 4x + 3$$

$$f'(x) = 4x + 3$$
$$f'(2) = 11$$

3. (5 points) Use differentiation rules to confirm your derivative above. Then find an equation of the line tangent to the graph of $f(x) = 2x^2 + 3x$ at the point where $x = 2$.

$f'(x) = 4x + 3$

Slope = $f'(2) = 11$

Point: $x = 2$, $y = 14$

$(2, 14)$

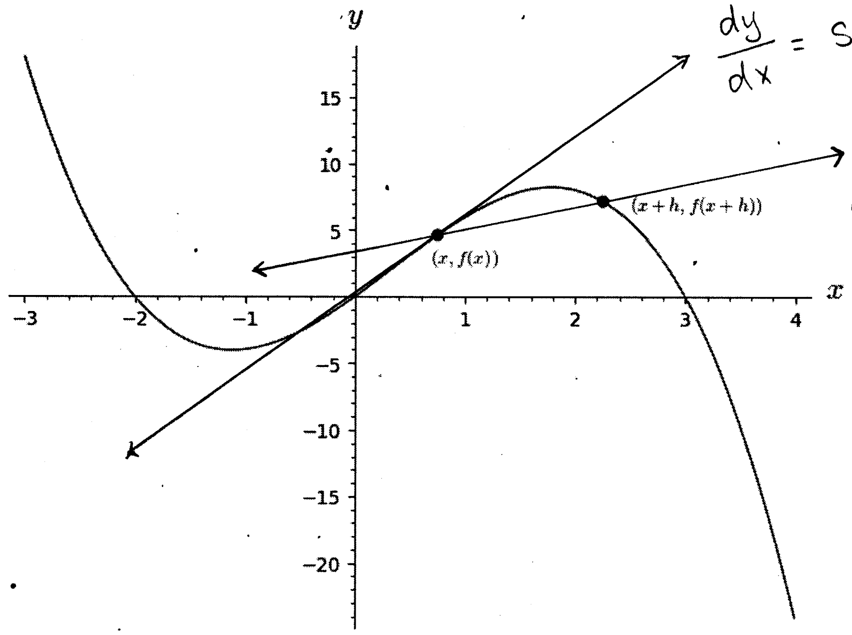
Tangent Line:

$y - 14 = 11(x - 2)$

or

$y = 11x - 8$

4. (5 points) The graph of $y = f(x)$ is shown below. Use the graph and the labeled points to illustrate the difference between $\frac{dy}{dx}$ and $\frac{\Delta y}{\Delta x}$. Explain your reasoning if necessary.



$\frac{dy}{dx}$ = SLOPE OF TANGENT LINE AT X

SLOPE OF SECANT LINE.

$= \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$

$(h = \Delta x)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

5. (5 points) Describe two ways a continuous function may fail to be differentiable.

① $f'(x)$ DNE IF THE GRAPH HAS A SHARP POINT AT X.

② $f'(x)$ DNE IF THE GRAPH HAS A VERTICAL TANGENT LINE AT X.

6. (6 points) A study found that a town's population (in thousands of people) can be modeled by $P(t) = -\frac{1}{3}t^3 + 64t + 3000$, where t is measured in years.

(a) Compute $P'(4)$. Give units on your result and interpret what the result means for the town.

$P'(t) = -t^2 + 64$

$P'(4) = -16 + 64 = \boxed{48 \text{ THOUSAND PEOPLE PER YEAR}}$

IN 4 YEARS, THE POPULATION WILL BE GROWING.

(b) Compute $P''(4)$. Give units on your result and interpret what the result means for the town's population.

$P''(t) = -2t$

$P''(4) = -8 = \boxed{-8 \text{ THOUSAND PEOPLE PER YEAR PER YEAR}}$

IN 4 YEARS, THE POPULATION GROWTH RATE WILL BE DECLINING. (GROWING BUT MORE SLOWLY.)

7. (6 points) Use trig identities and the quotient rule to derive our formula for the derivative of $y = \csc x$ from the basic rules for the sine and cosine.

$$\frac{d}{dx} \csc x = \frac{d}{dx} \frac{1}{\sin x} = \frac{(\sin x)(0) - (1)(\cos x)}{\sin^2 x} = -\frac{\cos x}{\sin^2 x}$$

$$= -\frac{1}{\sin x} \frac{\cos x}{\sin x} = -\csc x \cot x$$

8. (20 points) Determine the derivative of each function. Do not simplify.

(a) $y = x^2 + x + 1 + \sqrt[5]{x^2} + \frac{1}{x^5} = x^2 + x + 1 + x^{2/5} + x^{-5}$

$$\boxed{\frac{dy}{dx} = 2x + 1 + \frac{2}{5} x^{-3/5} - 5x^{-6}}$$

(b) $g(x) = (x^2 + 5x) \cot x$

$$\boxed{g'(x) = (2x+5) \cot x - (x^2 + 5x) \csc^2 x}$$

(c) $r(x) = \sin(x^4)$

$$\boxed{r'(x) = \cos(x^4)(4x^3) = 4x^3 \cos x^4}$$

(d) $y = \sqrt{100 - 2t - t^3} = (100 - 2t - t^3)^{1/2}$

$$\boxed{\frac{dy}{dt} = \frac{1}{2}(100 - 2t - t^3)^{-1/2}(-2 - 3t^2)}$$

9. An object is launched vertically upward from over the edge of a vertical cliff. The object's height (in feet) at after t seconds is given by

$$s(t) = -16t^2 + 96t + 256.$$

(a) (2 points) Determine the object's initial velocity and initial height. (Include units with your answers.)

$\boxed{V_0 = 96 \text{ FT}/\text{S}}$ $\boxed{S_0 = 256 \text{ FT}}$

WE CAN READ THESE RIGHT FROM $S(t)$.

(b) (2 points) Determine the average rate of change the object's height over the interval from $t = 1$ to $t = 2$. (Include units with your answer.)

$$\frac{\Delta S}{\Delta t} = \frac{S(2) - S(1)}{2-1} = \frac{384 - 336}{1} = \boxed{48 \text{ FT}/\text{S}}$$

(c) (2 points) Determine the function that gives the object's velocity at time t.

$$\boxed{V(t) = S'(t) = -32t + 96}$$

(d) (2 points) What is the acceleration of the object? (Include units with your answer.)

$$\boxed{a(t) = V'(t) = -32 \text{ FT}/\text{S}^2}$$

(e) (4 points) Determine the object's maximum height. (Include units with your answer.)

$V(t) = 0 \Rightarrow -32t + 96 = 0$

$\Rightarrow t = 3 \text{ s}$

$S(3) = -16(9) + 96(3) + 256$

$= \boxed{400 \text{ FT}}$

(f) (3 points) When does the object hit the ground?

$S(t) = 0 \Rightarrow -16(t^2 - 6t - 16) = 0$

$-16(t-8)(t+2) = 0$

$\boxed{t = 8 \text{ s}}$

(g) (2 points) What is the object's speed when it hits the ground?

$V(8) = -32(8) + 96 = -160 \text{ FT}/\text{S}$

$\boxed{\text{SPEED} = 160 \text{ FT}/\text{S}}$

10. (5 points) Let $F(x) = 3x^2 \cos x$. Find $F''(x)$.

$$F'(x) = 6x\cos x - 3x^2 \sin x$$

$$F''(x) = 6\cos x - 6x\sin x - 6x\sin x - 3x^2\cos x$$

OR

$$F''(x) = (6-3x^2)\cos x - 12x\sin x$$

11. (9 points) Find an equation of the line tangent to the graph of $y^3 + x^3 - xy = x + 5$ at the point $(2, 1)$.

$$3y^2\frac{dy}{dx} + 3x^2 - y - x\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1+y-3x^2}{3y^2-x}$$

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = \frac{1+1-12}{3-2} = \frac{-10}{1} = -10$$

Point: $(x,y) = (2,1)$

Tangent Line:

$$y - 1 = -10(x-2)$$

Follow-up: Find an equation of the normal line at $(2, 1)$.

$$m_\perp = \frac{1}{10}$$

$$y - 1 = \frac{1}{10}(x-2)$$