Math 131 - Final Exam

December 15, 2021

Name key

Score ______

Show all work to receive full credit. For each problem, place your final answer in the box provided. Each problem is worth 5 points–up to 2 points for the answer and up to 3 points for the supporting work or explanation.

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1. Determine the limit or briefly explain why it does not exist.

$$\lim_{x\to 2}\sqrt{4-x^2}$$

DOMAIN OF $f(x)=\sqrt{4-x^2}$ IS $[-2,2]$.

THIS LIMIT MAKES NO SENSE BECAUSE f IS NOT DEFINED ON BOTH SIDES OF $X=2$.

LIMIT DNE.	f IS NOT DEFINED IN AN INTERVAL AROUND $X=2$.

2. Determine the limit. Show analytically (not with a graph or table) how you got your answer.

$$\lim_{x\to 0}\frac{1+\cos x^3}{x(x+8)}$$

FORM $\frac{2}{0}$ ⇒ SOME KIND OF INF. LIMIT.

$$\lim_{X\to 0^+}\frac{1+\cos X^3}{X(X+8)}=+\infty$$

BECAUSE $\frac{1+\cos X^3}{X(X+8)}=\frac{+}{+}=+$ FOR $X\approx 0^+$

$$\lim_{X\to 0^-}\frac{1+\cos X^3}{X(X+8)}=-\infty$$

BECAUSE $\frac{1+\cos X^3}{X(X+8)}=\frac{+}{-}=-$ FOR $X\approx 0^-$

LIMIT DNE.

Form $\infty \cdot 0$

13. Evaluate the limit: $\lim_{x\to\infty} x^2e^{-x}$

$$= \lim_{x\to\infty} \frac{x^2}{e^x} \quad \frac{\infty}{\infty}$$

L'Hôpital's rule twice.

$$= \lim_{x\to\infty} \frac{2x}{e^x} \quad \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{2}{e^x} = \boxed{0}$$

$$\boxed{0}$$

14. Use calculus techniques to find the absolute extreme values of $f(x) = 3x^4 - 8x^3 - 48x^2$ on $[-3, 1]$.

$$f'(x) = 12x^3 - 24x^2 - 96x$$

$$= 12x(x^2 - 2x - 8)$$

$$= 12x(x-4)(x+2)$$

$$f'(x) = 0 \Rightarrow x = 0,\ \cancel{x = 4},\ x = -2$$

OUTSIDE INTERVAL!

x	f(x)
0	0
-2	-80 ← ABS MIN
-3	27 ← ABS MAX
1	-53

$$\boxed{\text{Abs min is } f(-2) = -80,\ \text{Abs max is } f(-3) = 27}$$

15. The second derivative of f is given by $f''(x) = (3x-1)(x+2)e^{15x}$. Find open intervals on which the graph of f is concave up.

$$f''(x) = 0 \Rightarrow x = \tfrac{1}{3},\ x = -2$$

Signs of f'': $+$ on $(-\infty, -2)$, $-$ on $(-2, \tfrac{1}{3})$, $+$ on $(\tfrac{1}{3}, \infty)$

$$\boxed{\text{Concave up on } (-\infty, -2) \cup (\tfrac{1}{3}, \infty)}$$

3. Determine the limit. Use algebraic techniques (not a graph, table, or L'Hôpital's rule) to show how you got your answer.

% More work

$$\lim_{x \to 2} \left[\frac{(x+2)^2 - 3x - 10}{x^2 - 2x} \right]$$

$$\lim_{x \to 2} \frac{x^2+4x+4-3x-10}{x(x-2)} = \lim_{x \to 2} \frac{x^2+x-6}{x(x-2)} = \lim_{x \to 2} \frac{(x-2)(x+3)}{x(x-2)}$$

$$= \lim_{x \to 2} \frac{x+3}{x} = \frac{5}{2}$$

$$\boxed{\frac{5}{2} = 2.5}$$

4. Yes or No: Is f continuous at $x = -2$? Use the definition of continuity to support your answer.

$$f(x) = \begin{cases} 2x + \cos \pi x, & x < -2 \\ x - 1, & x \geq -2 \end{cases}$$

$f(-2) = \boxed{-3}$

$$\lim_{x \to -2^-} f(x) = 2(-2) + \cos(-2\pi) = -4 + 1 = \boxed{-3}$$

$$\lim_{x \to -2^+} f(x) = -2 - 1 = \boxed{-3}$$

$$\boxed{\text{Yes. } f(-2) = \lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x)}$$

5. Find $\frac{d^2y}{dx^2}$ if $y = 5x^3 \sin x$.

$$\frac{dy}{dx} = 15x^2 \sin x + 5x^3 \cos x$$

$$\frac{d^2y}{dx^2} = 30x \sin x + 15x^2 \cos x + 15x^2 \cos x - 5x^3 \sin x$$

$$\boxed{(30x - 5x^3) \sin x + 30x^2 \cos x}$$

6. Let $f(x) = x^3$. Write $f'(x)$ in the box, then use the limit definition of derivative to obtain your answer.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2$$

$$= 3x^2 + 0 + 0$$

$$\boxed{f'(x) = 3x^2}$$

7. Find $f'(0)$ if $f(x) = (e^x + 2\cos x)^4$.

$$f'(x) = 4(e^x + 2\cos x)^3 (e^x - 2\sin x)$$

$$f'(0) = 4(1+2)^3(1-0) = 4(27)(1)$$

$$= 108$$

$$\boxed{108}$$

8. Find an equation of the line tangent to the graph of $x^2 + 2xy + y^4 = 5x - 1$ at the point $(2, 1)$.

$$2x + 2xy' + 2y + 4y^3y' = 5$$

$$2xy' + 4y^3y' = 5 - 2x - 2y$$

$$y' = \frac{5-2x-2y}{2x+4y^3}$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = \frac{5-4-2}{4+4} = -\frac{1}{8}$$

$$\boxed{y - 1 = -\frac{1}{8}(x-2) \quad \text{or} \quad y = -\frac{1}{8}x + \frac{5}{4}}$$

9. Let $g(x) = \sin^{-1}(2x) + \tan^{-1}(x^2) + \cos^{-1}(2x)$. Determine $g'(x)$.

$$g'(x) = \frac{2}{\sqrt{1-(2x)^2}} + \frac{2x}{1+(x^2)^2} - \frac{2}{\sqrt{1-(2x)^2}}$$

$$= \frac{2x}{1+x^4}$$

$$\boxed{g'(x) = \frac{2x}{1+x^4}}$$

10. Some values of $f(x)$ and $f'(x)$ near $x = 1$ are given in the table below.

x	0.50	0.75	1.00	1.25	1.50
$f(x)$	6.08	6.90	8.00	9.41	11.14
$f'(x)$	2.74	3.82	5.00	6.26	7.60

Determine the linearization of f at $x = 1$, and use it to approximate $f(0.9)$.

$$L(x) = f(1) + f'(1)(x-1) = 8 + 5(x-1) = 5x + 3$$

$$\boxed{L(x) = 5x + 3, \quad f(0.9) \approx L(0.9) = 7.5}$$

11. Find the slope of the line tangent to the graph of $y = \log_5(3x+2)^4$ at the point where $x = 1$. Write your answer in decimal form, rounded to the nearest thousandth.

$$y = \frac{4 \ln(3x+2)}{\ln 5}$$

$$\frac{dy}{dx} = \frac{4}{\ln 5} \cdot \frac{3}{3x+2} \qquad \left.\frac{dy}{dx}\right|_{x=1} = \frac{12}{5 \ln 5}$$

$$\boxed{\frac{12}{5 \ln 5} \approx 1.491}$$

12. Find the critical number(s) of $f(x) = \dfrac{x^2+4}{x}$.

$$f(x) = x + \frac{4}{x}, \quad f'(x) = 1 - \frac{4}{x^2}$$

$$f'(x) = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$$

$f'(x)$ DNE $\Rightarrow x = 0$ BUT $x = 0$ IS NOT IN THE DOMAIN OF f.

$$\boxed{x = \pm 2}$$

16. Find the function f for which $f'(x) = e^x + \sqrt{x} + \frac{2}{x}$ and $f(1) = 1$.

$$f(x) = \int \left(e^x + x^{1/2} + \frac{2}{x}\right) dx = e^x + \frac{2}{3}x^{3/2} + 2\ln|x| + C$$

$$f(1) = 1 \Rightarrow e^1 + \frac{2}{3} + 0 + C = 1$$

$$C = \frac{1}{3} - e$$

$$\boxed{f(x) = e^x + \frac{2}{3}x^{3/2} + 2\ln|x| + \frac{1}{3} - e}$$

17. Let $f(x) = 2x$. Use 6 subintervals of equal length and subinterval midpoints (for the c_k's) to compute a Riemann sum for f on $[1, 4]$.

$$\Delta x = \frac{4-1}{6} = \frac{1}{2}$$

PARTITION IS $1 < 1.5 < 2 < 2.5 < 3 < 3.5 < 4$

1.25 1.75 2.25 2.75 3.25 3.75

RIEMANN SUM IS $\frac{1}{2}\left[2.5 + 3.5 + 4.5 + 5.5 + 6.5 + 7.5\right]$

$= 15$

$$\boxed{15}$$

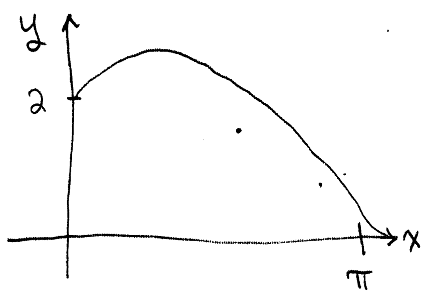
18. Evaluate the definite integral: $\displaystyle\int_0^2 (2x^3 - 7x^2 + 4x - 1)\,dx$

$$= \frac{1}{2}x^4 - \frac{7}{3}x^3 + 2x^2 - x \Big|_0^2$$

$$= \left(8 - \frac{56}{3} + 8 - 2\right) - (0) = -\frac{14}{3}$$

$$\boxed{-\frac{14}{3}}$$

19. Find the area of the region between the graph of $y = 1 + \sin x + \cos x$ and the x-axis over the interval $[0, \pi]$.



$$\int_0^{\pi} (1 + \sin x + \cos x)\,dx$$

$$= x - \cos x + \sin x \Big|_0^{\pi}$$

$$= (\pi - \cos\pi + \sin\pi) - (0 - 1 + 0)$$

$$= \pi + 2$$

$$\boxed{\pi + 2}$$

20. Use a u-substitution to evaluate the indefinite integral: $\displaystyle\int x(4x^2 - 1)^5\,dx$

$$\frac{1}{8}\int u^5\,du$$

$$= \frac{1}{48}u^6 + C$$

$u = 4x^2 - 1$

$du = 8x\,dx$

$\frac{1}{8}du = x\,dx$

$$\boxed{\frac{1}{48}(4x^2-1)^6 + C}$$