Math 131 - Test 1

September 14, 2022

Name key

Score ____________

Show all work to receive full credit. Supply explanations where necessary. You may get partial credit on multiple choice problems if you supply correct work or explanations.

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1. (6 points) Suppose that $\lim_{x\to 2} f(x) = 3$, $\lim_{x\to 2} g(x) = 7$, and $\lim_{x\to 2} h(x)$ exists.

   (a) Evaluate $\lim_{x\to 2}[5\,f(x) - x\,g(x)]$.

   $$5\lim_{x\to 2} f(x) - \left(\lim_{x\to 2} x \lim_{x\to 2} g(x)\right) = 5(3) - 2(7) = \boxed{1}$$

   (b) Find $\lim_{x\to 2} h(x)$ if $\lim_{x\to 2} \dfrac{g(x)}{h(x)} = \dfrac{1}{2}$.

   $$\frac{\lim_{x\to 2} g(x)}{\lim_{x\to 2} h(x)} = \frac{1}{2} \Rightarrow \frac{7}{\lim_{x\to 2} h(x)} = \frac{1}{2} \Rightarrow \lim_{x\to 2} h(x) = \boxed{14}$$

   (c) Find $\lim_{x\to 2} h(x)$ if $\lim_{x\to 2} \dfrac{f(x)}{h(x)}$ does not exist.

   IF $\lim_{x\to 2} h(x)$ EXISTS AND $\dfrac{\overbrace{\lim_{x\to 2} f(x)}^{3}}{\lim_{x\to 2} h(x)}$ DOES NOT, THEN

   $$\lim_{x\to 2} h(x) = \boxed{0}$$

   OTHERWISE, THE LIMIT WOULD EXIST!

2. (6 points) Use a table of numerical values to approximate the following limit. Your table must show function values at six or more points.

$$f(x) = \frac{5\tan(x-1)}{2x-2}$$

$$\lim_{x\to 1} \frac{5\tan(x-1)}{2x-2}$$

x	f(x)
0.9	2.50837
0.99	2.50008
0.999	2.50000
1.1	2.50837
1.01	2.50008
1.001	2.50000

IT LOOKS LIKE

$$\lim_{x\to 1} \frac{5\tan(x-1)}{2x-2} = 2.5.$$

3. (2 points) Which one of these statements is true about the limit concept?

(a) In order to have a limit at a point, a function must be defined at the point.
(b) A limit cannot exist if direct substitution fails.
(c) A function that is defined at a point must have a limit at that point.
d (d) A function's limit at a point and its value at the point are not necessarily related.

4. (2 points) Suppose $\lim_{x\to 9} f(x) = -5$. Which one of these statements is true?

(a) $f(9)$ must be equal to -5.
(b) $f(9)$ must be defined.
c (c) The domain of f must include some numbers greater than 9.
(d) The domain of f cannot include the number 9.

5. (2 points) Explain why this limit fails to exist: $\lim_{x\to 7} \frac{x+9}{(x-7)^2}$.

$\frac{16}{0^+}$ FORM

(a) Direct substitution results in division by zero.
(b) The limit from the left does not equal the limit from the right.
c (c) The function values grow without bound as the limit point is approached.
(d) The function is not defined on both sides of the limit point.

6. (2 points) Explain why this limit fails to exist.

$$\lim_{x\to -3} f(x) \quad \text{where} \quad f(x) = \begin{cases} 2x^3 + \cos(\pi x), & -3 \le x < 2 \\ x\sin(x), & x > 2 \end{cases}$$

NOT DEFINED TO THE LEFT OF $X = -3$

(a) The function values oscillate as the limit point is approached.
(b) The limit from the left does not equal the limit from the right.
(c) The function values grow without bound as the limit point is approached.
d (d) The function is not defined on both sides of the limit point.

7. (2 points) Explain why this limit fails to exist: $\lim_{x\to 5} \frac{2x-10}{|x-5|}$. $= \lim_{x\to 5} \frac{2(x-5)}{|x-5|}$

(a) Direct substitution results in division by zero.
b (b) The limit from the left does not equal the limit from the right.
(c) The function values grow without bound as the limit point is approached.
(d) The function is not defined on both sides of the limit point.

$\lim_{x\to 5^-} \frac{2(x-5)}{|x-5|} = -2$

$\lim_{x\to 5^+} \frac{2(x-5)}{|x-5|} = +2$

8. (24 points) Evaluate each limit analytically. Show all work.

(a) $\lim_{x\to -2} \left(\frac{x^3+6x^2+8x}{3x^2+5x-2}\right)$

0/0 MORE WORK

$$= \lim_{x\to -2} \frac{\cancel{(x+2)}\, x\,(x+4)}{\cancel{(x+2)}(3x-1)} = \frac{-2(2)}{-7} = \boxed{\frac{4}{7}}$$

(b) $\lim_{x\to 0} \frac{\sin(6x)}{7x}$

0/0 MORE WORK

$$= \lim_{x\to 0} \frac{6}{6}\,\frac{\sin 6x}{7x} = \lim_{x\to 0} \frac{6}{7}\,\frac{\sin 6x}{6x}$$

$$= \left(\frac{6}{7}\right)(1) = \boxed{\frac{6}{7}}$$

(c) $\lim_{h\to 3} \frac{\sqrt{h}-\sqrt{3}}{2h-6}$

0/0 MORE WORK

$$\lim_{h\to 3} \frac{\sqrt{h}-\sqrt{3}}{2(h-3)} \cdot \frac{\sqrt{h}+\sqrt{3}}{\sqrt{h}+\sqrt{3}} = \lim_{h\to 3} \frac{\cancel{h-3}}{2\cancel{(h-3)}(\sqrt{h}+\sqrt{3})}$$

$$= \frac{1}{2(\sqrt{3}+\sqrt{3})} = \boxed{\frac{1}{4\sqrt{3}}}$$

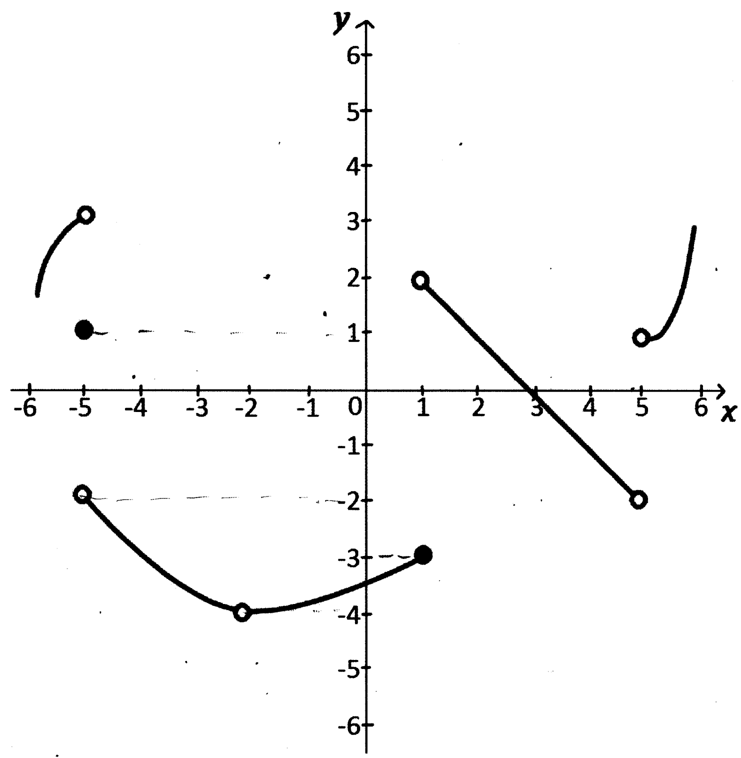
(d) $\lim_{t\to 1^-} \left[\frac{(t-2)^2-1}{t-1}\right]$

0/0 MORE WORK

$$= \lim_{t\to 1^-} \frac{t^2-4t+4-1}{t-1} = \lim_{t\to 1^-} \frac{t^2-4t+3}{t-1}$$

$$= \lim_{t\to 1^-} \frac{\cancel{(t-1)}(t-3)}{\cancel{t-1}} = \boxed{-2}$$

9. (14 points) The graph of $y = f(x)$ is shown below. Use the graph to solve each part of this problem.



(a) What type of discontinuity does f have at $x = -2$? Explain your reasoning.

REMOVABLE, BECAUSE THE LIMIT EXISTS AT $X = -2$

(b) Estimate $\lim_{x \to -5^+} f(x)$. $\approx$ $\boxed{-2}$

(c) Estimate the value $f(-5)$. $\approx$ $\boxed{1}$

(d) Estimate $\lim_{x \to -2} f(x)$. $\approx$ $\boxed{-4}$

(e) Explain why $\lim_{x \to -6} f(x)$ does not exist. THE GRAPH STOPS AT -6!

f IS NOT DEFINED FOR $X < -6$.

(f) What type of discontinuity does f have at $x = 5$? Explain your reasoning.

Jump. LIMIT FROM LEFT $\neq$ LIMIT FROM RIGHT.

(g) Estimate $\lim_{x \to 1^-} f(x)$. $\approx$ $\boxed{-3}$

10. (8 points) Consider the rational function $Q(x) = \dfrac{x^2+4x+4}{x^2+3x+2}$.

(a) Determine the discontinuities of Q and briefly explain your reasoning.

$$Q(x) = \frac{(x+2)(x+2)}{(x+2)(x+1)}$$

THE ONLY DISCONTINUITIES OCCUR WHERE THE DENOM IS ZERO: $\boxed{x=-1,\ x=-2}$

(b) Classify the discontinuities you found in part (a).

$x=-2$ IS A REMOVABLE DISCONT BECAUSE THE LIMIT EXISTS:

$$\lim_{x\to -2} Q(x) = \lim_{x\to -2} \frac{x+2}{x+1} = 0$$

$x=-1$ IS AN INFINITE DISCONT BECAUSE

$$Q(x) = \frac{x+2}{x+1}, \quad x \neq -2$$

AND SUBSTITUTION OF $x=-1$ YIELDS $\frac{k\neq 0}{0}$ FORM.

(c) Determine the vertical asymptotes of the graph of Q.

SEE PART (b).

$\boxed{\text{THE ONLY V.A. IS } x=-1.}$

11. (8 points) Find the number b so that f is continuous everywhere. Be sure to show how you have used the definition of continuity.

$$f(x) = \begin{cases} x^2+bx+5, & x<-1 \\ 2x-3, & x \geq -1 \end{cases}$$

EACH "PIECE" IS CONTINUOUS EVERYWHERE, SO THE ONLY POSSIBLE DISCONT. IS AT $x=-1$.

$$f(-1) = -5 = \lim_{x\to -1^+} f(x)$$

$$\lim_{x\to -1^-} f(x) = (-1)^2 + b(-1) + 5 = 6-b$$

$$6-b=-5$$

$$\Rightarrow \boxed{b=11}$$

12. (5 points) Suppose that f is a function satisfying the inequalities

$$\underbrace{-\frac{1}{2}x^2 + 2x - 1}_{g(x)} \leq f(x) \leq \underbrace{\frac{1}{24}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{2}{3}x - \frac{1}{3}}_{h(x)}$$

for $0 \leq x \leq 4$. What can be said about $\lim_{x\to 2} f(x)$? Explain your reasoning.

$\lim_{x\to 2} g(x) = g(2) = -\frac{1}{2}(4) + 4 - 1 = 1$

$\lim_{x\to 2} h(x) = h(2) = \frac{1}{24}(16) - \frac{1}{3}(8) + \frac{1}{2}(4) + \frac{2}{3}(2) - \frac{1}{3} = 1$

$\lim_{x\to 2} g(x) \leq \lim_{x\to 2} f(x) \leq \lim_{x\to 2} h(x)$

$\Downarrow$

$\boxed{\lim_{x\to 2} f(x) = 1 \text{ BY SQUEEZE THM}}$

13. (9 points) In each problem below, determine analytically whether the limit is $+\infty$, $-\infty$, or DNE. Show work or explain your reasoning.

(a) $\lim_{x\to 1} \frac{8}{(x-1)^3}$

$8/0 \Rightarrow$ SOME KIND OF INF. LIMITS

TO THE LEFT OF $X=1$:

$\frac{8}{(X-1)^3} = \frac{+}{-} = -$

LIMIT FROM LEFT IS $-\infty$.

TO THE RIGHT OF $X=1$:

$\frac{8}{(X-1)^3} = \frac{+}{+} = +$

LIMIT FROM RIGHT IS $+\infty$.

CONCLUSION: LIMIT DNE

(b) $\lim_{x\to 7^-} \left(\frac{3-x}{x-7}\right)$

$-4/0 \Rightarrow$ SOME KIND OF INF. LIMIT.

TO THE LEFT OF $X=7$:

$\frac{3-X}{X-7} = \frac{-}{-} = +$

$\boxed{\text{LIMIT IS } +\infty}$

(c) $\lim_{x\to 4} \left(\frac{2x+4}{(x-4)^2}\right)$

$12/0 \Rightarrow$ SOME KIND OF INF. LIMITS.

TO THE LEFT OF $X=4$:

$\frac{2X+4}{(X-4)^2} = \frac{+}{+} = +$

LIMIT FROM LEFT IS $+\infty$.

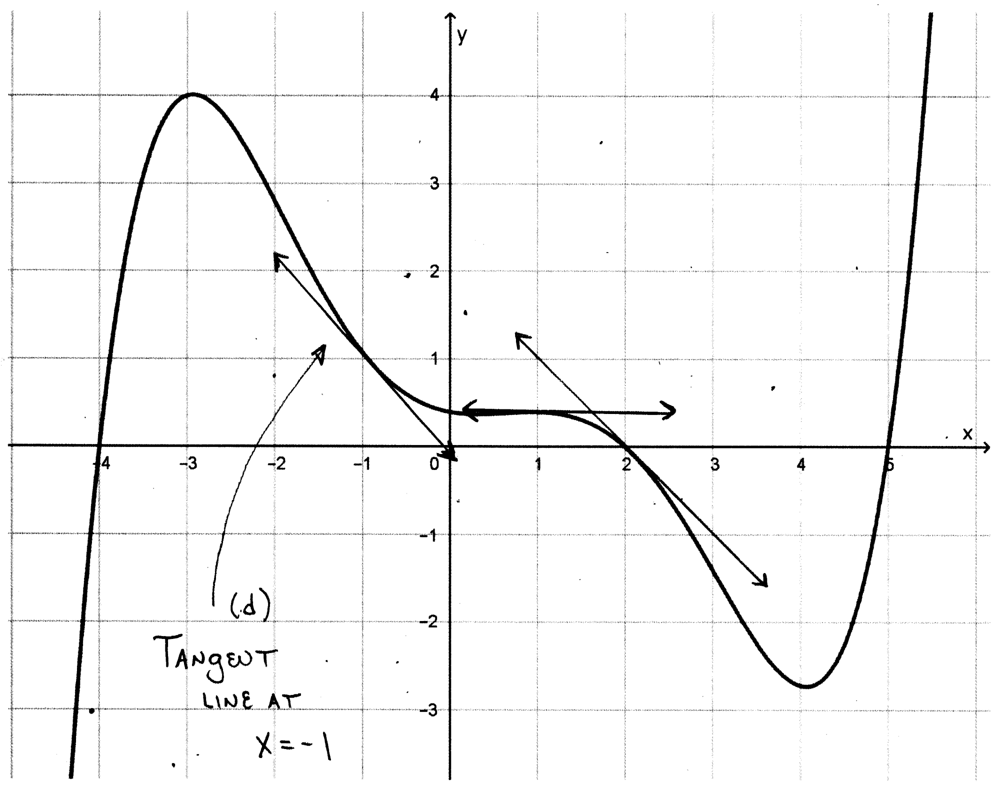
TO THE RIGHT OF $X=4$:

$\frac{2X+4}{(X-4)^2} = \frac{+}{+} = +$

LIMIT FROM THE RIGHT IS $+\infty$.

CONCLUSION: LIMIT IS $+\infty$

14. (10 points) The graph of $y = f(x)$ is shown below. Use the graph to solve each part of this problem. Explain your reasoning for each part.



(a) Estimate the value of the derivative at $x = 1$.

$f'(0) \approx 0$ TANGENT LINE IS HORIZONTAL (OR CLOSE TO HORIZONTAL).

(b) Estimate the value of the derivative at $x = 2$.

$f'(2) \approx -1$ THE SLOPE OF THE TANGENT LINE AT $X = 2$ IS ABOUT -1. THE LINE I'VE DRAWN PASSES THROUGH $(2,0)$ & $(3,-1)$.

(c) Determine an x-value at which the derivative is a rather big positive number.

$X = -4$. GRAPHS STEEPLY RISES AT $X = -4$

(d) On the graph above, sketch the tangent line at $x = -1$.

SEE ABOVE.