**Math 131 - Test 3**
November 9, 2022

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. Unless otherwise indicated, use differentiation rules for all derivatives and do not simplify.

1. (6 points) Find the slope of the line tangent to the graph of $y = \dfrac{\ln(x^2)}{x^2}$ at the point where $x = e$.

$$y = \frac{2\ln x}{x^2}$$

$$\frac{dy}{dx} = \frac{(x^2)\left(\frac{2}{x}\right) - (2\ln x)(2x)}{x^4}$$

$$\frac{dy}{dx} = \frac{2x - 4x\ln x}{x^4} = \frac{2 - 4\ln x}{x^3}$$

$$\left.\frac{dy}{dx}\right|_{x=e} = \boxed{\frac{-2}{e^3} \approx -0.09957}$$

2. (6 points) Compute $f'(4)$ when $f(x) = 3^{2-x}$.

$$f'(x) = 3^{2-x}(-1)(\ln 3)$$

$$f'(4) = 3^{-2}(-\ln 3) = \boxed{-\frac{\ln 3}{9} \approx -0.1221}$$

Follow-up: Use your answer to determine whether $f$ is increasing, decreasing, or neither at the point where $x = 4$.

$f'(4)$ IS NEGATIVE $\Rightarrow$ $\boxed{f \text{ IS DECREASING AT } x=4.}$

3. (10 points) For $x > 1$, let $y = \dfrac{x(x-1)^{3/2}}{\sqrt{x+1}}$. Use logarithmic differentiation to find $dy/dx$.

$$\ln y = \ln x + \frac{3}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{3}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x+1}$$

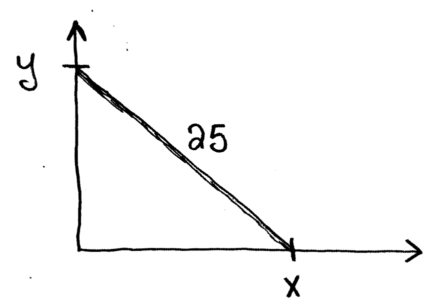
$$\boxed{\frac{dy}{dx} = \left(\frac{x(x-1)^{3/2}}{\sqrt{x+1}}\right)\left(\frac{1}{x} + \frac{3}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x+1}\right)}$$

4. (4 points) Suppose $x$ and $y$ are differentiable functions of $t$ and that $y = 4\pi x^2$. Find a formula that relates $dy/dt$ and $dx/dt$.

$$y = 4\pi x^2 \Rightarrow \frac{dy}{dt} = 4\pi\left(2x\frac{dx}{dt}\right)$$

$$\boxed{\frac{dy}{dt} = 8\pi x\frac{dx}{dt}}$$

5. (6 points) A 25-ft long ladder is leaning against a vertical wall. The base of the ladder is pulled away from the wall at a constant rate of 2 ft/sec. At what rate is the top of the ladder sliding down the wall at the moment the base of the ladder is 7 ft from the wall?




$$\frac{dx}{dt} = 2$$

Find $\frac{dy}{dt}$ when $x = 7$.

$$x^2 + y^2 = (25)^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-2x\frac{dx}{dt}}{2y}$$

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

When $x = 7$,

$$y^2 = 625 - 49$$

$$= 576$$

or $y = 24$

- and -

$$\frac{dy}{dt} = -\frac{7}{24}(2)$$

$$= \boxed{-\frac{7}{12}} \text{ ft/sec}$$

6. (5 points) Given the following information, find and use the linearization of $f$ at $x = 3$ to approximate $f(2.98)$.

$$f(2) = 4.21, \quad f'(2) = 0.44, \quad f(3) = 4.90, \quad f'(3) = 1.67$$

$L(x) = f(3) + f'(3)(x-3)$

$\boxed{L(x) = 4.90 + 1.67(x-3)}$

$f(2.98) \approx L(2.98)$

$= 4.9 + 1.67(-0.02)$

$= \boxed{4.8666}$

7. (5 points) Determine the differential $dy$ when $y = e^{2x}\sin x$.

$\boxed{dy = \left(2e^{2x}\sin x + e^{2x}\cos x\right)dx}$

8. (8 points) Determine the critical numbers of $g(x) = 3x - 10x^{3/5}$.

g IS DEFINED FOR ALL REAL NUMBERS.

$g'(x) = 3 - 6x^{-2/5}$

$g'(x)$ DNE WHEN $X = 0$

$g'(x) = 0 \Rightarrow 6x^{-2/5} = 3$

$x^{-2/5} = \frac{1}{2} \Rightarrow x^{2/5} = 2$

$x^2 = 32$

$x = \pm\sqrt{32} = \pm 2\sqrt{2}$

CRITICAL #'S ARE

$X = 0, \quad X = 4\sqrt{2}, \quad X = -4\sqrt{2}$

9. (10 points) Use calculus techniques to find the absolute extreme values of $f(x) = 3x^4 - 4x^3$ on $[-1, 2]$.

$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$

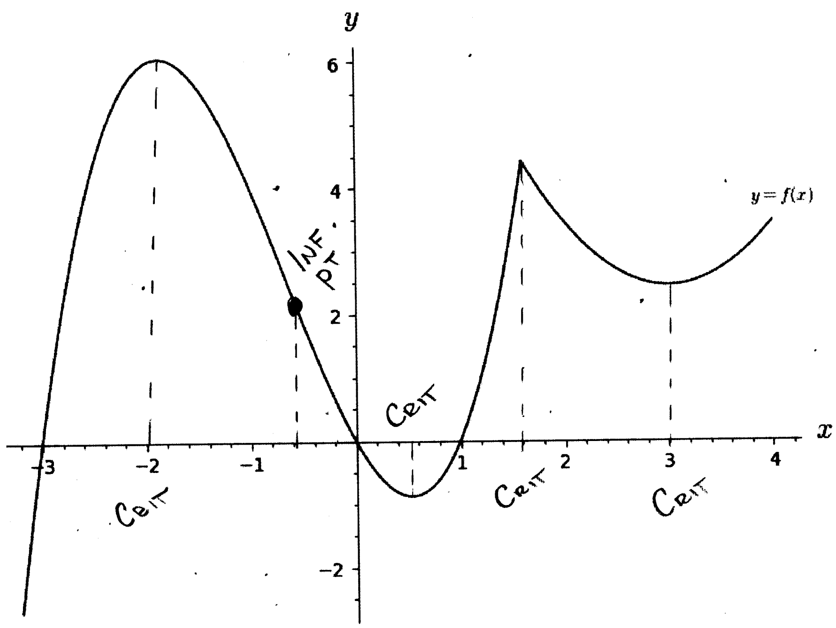
$f'(x) = 0 \Rightarrow X = 0$ OR $X = 1$

CRIT #'S : $X = 0, X = 1$

ENDPTS : $X = -1, X = 2$

| X | f(x) | |
|---|---|---|
| 0 | 0 | |
| 1 | $\boxed{-1}$ | ← ABS MIN |
| -1 | 7 | |
| 2 | $\boxed{16}$ | ← ABS MAX |

10. (8 points) The graph of the function $f$ is shown below. Use the graph to solve each part of this problem.




(a) Estimate the critical numbers of $f$.

$X = -2, \quad X = \frac{1}{2}, \quad X = 1.6, \quad X = 3$

(b) Determine open intervals on which $f$ is increasing/decreasing.

INCREASING ON $(-\infty, -2) \cup (0.5, 1.6) \cup (3, \infty)$

DECREASING ON $(-2, 0.5) \cup (1.6, 3)$

(c) Estimate the locations of all inflection points.

$X = -0.6$

(d) Determine open intervals on which the graph is concave up/down.

CONCAVE UP ON $(-0.6, 1.6) \cup (1.6, \infty)$

CONCAVE DOWN ON $(-\infty, -0.6)$

11. (6 points) Use the 2nd derivative to determine whether the graph of $y = x^3 + \sin(10x)$ is concave up or concave down at the point where $x = 0.65$.

$$\frac{dy}{dx} = 3x^2 + 10\cos(10x)$$

$$\frac{d^2y}{dx^2} = 6x - 100\sin(10x)$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0.65} = 6(0.65) - 100\sin(6.5) \approx -17.612$$

$\boxed{\text{GRAPH IS CD.}}$

12. (4 points) Explain why $x = 0$ is not a critical number of $f(x) = \sqrt{x}$ even though $f'(0)$ is not defined.

X = 0 IS A DOMAIN ENDPOINT.

IT IS NOT A DOMAIN INTERIOR POINT.

13. (6 points) The function $f$ has critical numbers $x = -3$, $x = 0$, and $x = 2$. Use the information given below to determine whether each critical number gives a relative maximum, a relative minimum, or neither.

$$f''(-3) = 0, \quad \overset{\text{CD}}{f''(0) = -2}, \quad \overset{\text{CD}}{f''(2) = -1}, \quad f'(-4) = -12, \quad f'(-1) = 5, \quad f'(1) = -15$$

$\Downarrow$ 2ND DERIV. TEST IS INCONCLUSIVE AT X = -3

$\Downarrow$ $\boxed{\text{X = 0 GIVES A REL. MAX.}}$

$\Downarrow$ $\boxed{\text{X = 2 GIVE A REL. MAX.}}$

USE 1ST DERIV.

SIGNS OF $f'(x)$: $-$ | $+$ | $-$ | ?

X = -3    X = 0    X = 2

$\Downarrow$

$\boxed{\text{X = -3 GIVES A REL. MIN.}}$

14. (16 points) Let $f(x) = x^5 - 10x^4 + 25x^3$. Apply the 1st and 2nd derivative tests to find open intervals on which $f$ is increasing/decreasing and open intervals on which the graph of $f$ is concave up/down. Also identify all relative extreme values and inflection points. (You will find it useful to know that the solutions of $2x^2 - 12x + 15 = 0$ are $x = \frac{6-\sqrt{6}}{2} \approx 1.775$ and $x = \frac{6+\sqrt{6}}{2} \approx 4.225$.)

$$f'(x) = 5x^4 - 40x^3 + 75x^2$$
$$= 5x^2(x^2 - 8x + 15)$$
$$= 5x^2(x-3)(x-5)$$

CRIT #'S ARE $X = 0$, $X = 3$, $X = 5$

SIGNS OF $f'$: $+$ | $+$ | $-$ | $+$

| | | 0 | | 3 | | 5 | |
|---|---|---|---|---|---|---|---|
| $f$ IS | INC | | INC | | DEC | | INC |

$$f''(x) = 20x^3 - 120x^2 + 150x$$
$$= 10x(2x^2 - 12x + 15)$$

PIPS ARE $X = 0$, $X = \frac{6-\sqrt{6}}{2}$, $X = \frac{6+\sqrt{6}}{2}$

SIGNS OF $f''$: $-$ | $+$ | $-$ | $+$

| | | 0 | | $\frac{6-\sqrt{6}}{2}$ | | $\frac{6+\sqrt{6}}{2}$ | |
|---|---|---|---|---|---|---|---|
| GRAPH IS | CD | | CU | | CD | | CU |

$f$ IS INCREASING ON $(-\infty, 0) \cup (0,3) \cup (5, \infty)$.

$f$ IS DECREASING ON $(3,5)$.

$f(3) = 108$ IS A REL. MAX

$f(5) = 0$ IS A REL. MIN.

GRAPH IS CU ON $\left(0, \frac{6-\sqrt{6}}{2}\right) \cup \left(\frac{6+\sqrt{6}}{2}, \infty\right)$

GRAPH IS CD ON $(-\infty, 0) \cup \left(\frac{6-\sqrt{6}}{2}, \frac{6+\sqrt{6}}{2}\right)$.

INF. PTS. ARE $(0,0)$, $\left(\frac{6-\sqrt{6}}{2}, 58.18\right)$, $\left(\frac{6+\sqrt{6}}{2}, 45.32\right)$