Math 131 - Final Exam

December 14, 2022

Name key

Score ____________

Show all work to receive full credit. For each problem, place your final answer in the box provided. Each problem is worth 5 points–up to 2 points for the answer and up to 3 points for the supporting work or explanation. Derivatives need not be simplified.

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1. Use a table of numerical values to estimate the limit: $\lim_{x \to -6^+} \frac{\sqrt{10-x}-4}{x+6}$

$f(x) = \frac{\sqrt{10-x}-4}{x+6}$

x	-5.9	-5.99	-5.999	-5.9999
$f(x)$	-0.125196	-0.12502	-0.125	-0.125

$\boxed{\text{It looks like } \lim_{x \to -6^+} f(x) = -0.125}$

2. Determine the limit. Use algebraic techniques (not a graph, table, or L'Hôpital's rule) to show how you got your answer.

$$\lim_{x \to 2} \frac{\frac{1}{2} - \frac{1}{x}}{x-2}$$

0/0 More work!

$$\lim_{x \to 2} \frac{\frac{x}{2x} - \frac{2}{2x}}{x-2} = \lim_{x \to 2} \frac{\cancel{x-2}}{2x} \cdot \frac{1}{\cancel{x-2}} = \lim_{x \to 2} \frac{1}{2x}$$

$$= \frac{1}{4}$$

$\boxed{\frac{1}{4}}$

3. Find the number b so that the function f is continuous everywhere. Make sure your work shows how you used the definition of continuity.

$$f(x) = \begin{cases} 5x^3 + bx + \cos(\pi x), & x < -1 \\ 3x - 9, & x \geq -1 \end{cases}$$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \left(5x^3 + bx + \cos(\pi x)\right)$$

$$= 5(-1) + b(-1) + \cos(-\pi)$$

$$= -6 - b = f(-1) = 3(-1) - 9$$

$$-6 - b = -12$$

$$\boxed{b = 6}$$

4. The function $g(x) = \dfrac{|x-5|}{x^2 - 4x - 5}$ has two discontinuities. Find and classify them.

$$g(x) = \frac{|x-5|}{(x-5)(x+1)}$$

THERE ARE TWO DISCONTS: $X = 5$ AND $X = -1$.

$X = -1$ gives A $\frac{6}{0}$ FORM $\Rightarrow$ $X = -1$ IS AN INF. DISCONT.

$X = 5$:

$$\left.\begin{aligned} \lim_{x \to 5^+} \frac{|x-5|}{(x-5)(x+1)} &= \lim_{x \to 5^+} \frac{1}{x+1} = \frac{1}{6} \\ \lim_{x \to 5^-} \frac{|x-5|}{(x-5)(x+1)} &= \lim_{x \to 5^-} \frac{-1}{x+1} = -\frac{1}{6} \end{aligned}\right\}$$ $X = 5$ IS A jump DISCONT.

$X = -1$ IS AN INFINITE DISCONTINUITY.

$X = 5$ IS A jump DISCONTINUITY.

5. Let $f(x) = 3x - x^2$. Write $f'(x)$ in the box, then use the limit definition of derivative to obtain your answer.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[3(x+h) - (x+h)^2] - [3x - x^2]}{h}$$

$$= \lim_{h\to 0} \frac{3x + 3h - x^2 - 2xh - h^2 - 3x + x^2}{h}$$

$$= \lim_{h\to 0} \frac{3h - 2xh - h^2}{h} = \lim_{h\to 0} (3 - 2x - h)$$

$$= 3 - 2x$$

$$\boxed{f'(x) = 3 - 2x}$$

6. Find an equation of the line tangent to the graph of $f(x) = \dfrac{15}{\sqrt[3]{x}}$ at the point where $x = -8$. Write your answer in the form $y = mx + b$.

$$f'(x) = \frac{d}{dx} 15x^{-1/3} = -5x^{-4/3}$$

$$= \frac{-5}{\sqrt[3]{x^4}}$$

$$m = f'(-8) = -\frac{5}{16}$$

$$f(-8) = \frac{15}{-2}$$

$$y + \frac{15}{2} = -\frac{5}{16}(x + 8)$$

$$y + \frac{15}{2} = -\frac{5}{16}x - \frac{5}{2}$$

$$\boxed{y = -\frac{5}{16}x - 10}$$

7. Let $f(x) = \dfrac{\sin x}{x}$. Determine $f''(x)$.

$$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

$$f''(x) = \frac{x^2\left[\cancel{\cos x} - x\sin x - \cancel{\cos x}\right] - (x\cos x - \sin x)(2x)}{x^4}$$

$$= \frac{-x^2\sin x - 2x\cos x + 2\sin x}{x^3}$$

$$\boxed{f''(x) = \frac{2\sin x - 2x\cos x - x^2\sin x}{x^3}}$$

8. Compute the derivative: $\dfrac{d}{dx}\tan^5 x$. $= \dfrac{d}{dx}(\tan x)^5$

Chain Rule

$$= 5\tan^4 x \cdot \sec^2 x$$

$$\boxed{5\tan^4 x \cdot \sec^2 x}$$

9. An object is falling in such a way that its height (in feet) after t seconds is given by

$$s(t) = -16t^2 + 64t + 80.$$

What is the object's speed when it hits the ground?

$S(t) = 0 \Rightarrow -16(t^2 - 4t - 5) = 0$

$(t-5)(t+1) = 0 \Rightarrow t = 5$

$S'(t) = -32t + 64$

$|S'(5)| = |-32(5) + 64|$

$= 96$

$$\boxed{96 \text{ FT}/\text{S}}$$

10. Let $h(x) = \sin^{-1}(f(x))$. Given the following information, compute $h'(3)$.

$$f(1) = \frac{1}{3}, \quad f'(1) = \frac{\sqrt{5}}{2}, \quad f(3) = \frac{\sqrt{3}}{2}, \quad f'(3) = \frac{1}{2}$$

CHAIN RULE...

$$h'(x) = \frac{f'(x)}{\sqrt{1 - f(x)^2}} \qquad h'(3) = \frac{f'(3)}{\sqrt{1 - f(3)^2}}$$

$$= \frac{1/2}{\sqrt{1 - \frac{3}{4}}} = \frac{1/2}{1/2} = 1$$

$$\boxed{h'(3) = 1}$$

11. Find $\frac{dy}{dx}$ if $y = \log_3[x^2(9x-7)]$.

$$y = \frac{1}{\ln 3}\left(2\ln x + \ln(9x-7)\right)$$

$$\frac{dy}{dx} = \frac{1}{\ln 3}\left(\frac{2}{x} + \frac{9}{9x-7}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\ln 3}\left(\frac{2}{x} + \frac{9}{9x-7}\right)}$$

12. Find the linearization of $f(x) = x\tan x$ at $x = \pi/4$. Then use your linearization to approximate $f(0.75)$.

$$f\left(\frac{\pi}{4}\right) = \frac{\pi}{4}\tan\frac{\pi}{4} = \frac{\pi}{4}$$

$$f'(x) = x\sec^2 x + \tan x$$

$$f'\left(\frac{\pi}{4}\right) = \frac{\pi}{4}(2) + 1 = \frac{\pi}{2}+1$$

$$\boxed{L(x) = \frac{\pi}{4} + \left(\frac{\pi}{2}+1\right)\left(x-\frac{\pi}{4}\right)}$$

$$L(0.75) = \frac{\pi}{4} + \left(\frac{\pi}{2}+1\right)\left(0.75-\frac{\pi}{4}\right)$$

$$\approx 0.6944$$

$$\boxed{f(0.75) \approx L(0.75) \approx 0.6944}$$

13. Use calculus techniques to find the absolute extreme values of $g(x) = 8x - x^4$ on the interval $[-2, 1]$.

$f'(x) = 8x - x^4$

$f'(x) = 8 - 4x^3$

$f'(x) = 0 \Rightarrow x^3 = 2$

$\Rightarrow x = \sqrt[3]{2} \approx 1.26$

NOT A CRIT #.

NO CRIT #'S.

END PTS: $x = -2$, $x = 1$

x	$g(x)$	
-2	-32	← ABS MIN
1	7	← ABS MAX

$\boxed{\text{ABS MAX} = g(1) = 7, \ \text{ABS MIN} = g(-2) = -32}$

14. The function f is differentiable for all x-values, and f has exactly two critical points: $x = 2$ and $x = 5$. Furthermore, $f(2)$ is a relative minimum and $f(5)$ is neither a relative minimum nor maximum. Determine the sign (positive or negative) of the expression $f'(-1) \cdot f'(3) \cdot f'(99)$.

1ST DERIVATIVE TEST

SIGNS OF f': $-$ | $x = 2$ | $+$ | $x = 5$ | $+$

f IS: DEC | $x = 2$ | INC | $x = 5$ | INC

$f'(-1) \cdot f'(3) \cdot f'(99)$

$=$ NEG $\times$ POS $\times$ POS $=$ NEG

$\boxed{\text{NEGATIVE.}}$

15. Evaluate the limit: $\lim_{x \to 1} \dfrac{\int_1^x \cos\theta \, d\theta}{x - 1}$

0/0 MORE WORK

L'HÔPITAL'S RULE ...

$$\lim_{x \to 1} \frac{\cos x}{1} = \cos(1)$$

$\boxed{\text{LIMIT IS } \cos(1) \approx 0.5403}$

16. Suppose you know that $\int_0^3 2x^2 \, dx = A$ and $\int_0^3 x \, dx = B$. What is the value of $\int_3^0 (3x^2 - 7x) \, dx$ in terms of A and B?

$$\int_3^0 (3x^2 - 7x) \, dx = -\int_0^3 (3x^2 - 7x) \, dx = -\left[\frac{3}{2}\int_0^3 2x^2 dx - 7\int_0^3 x \, dx\right]$$

$$= -\left[\frac{3}{2}A - 7B\right]$$

$\boxed{-\frac{3}{2}A + 7B}$

17. Let $f(x) = 32x^2 - 8x$. Use four subintervals of equal length and left subinterval endpoints to compute the corresponding (left) Riemann sum for f on $[1, 2]$.

$$\Delta x = \frac{2-1}{4} = \frac{1}{4}$$

$$c_1 = 1$$
$$c_2 = \frac{5}{4}$$
$$c_3 = \frac{6}{4}$$
$$c_4 = \frac{7}{4}$$

RIEMANN SUM =

$$f(1)\left(\tfrac{1}{4}\right) + f\left(\tfrac{5}{4}\right)\left(\tfrac{1}{4}\right) + f\left(\tfrac{6}{4}\right)\left(\tfrac{1}{4}\right) + f\left(\tfrac{7}{4}\right)\left(\tfrac{1}{4}\right) = 52$$

$$\boxed{52}$$

18. Evaluate the definite integral: $\displaystyle\int_1^3 \left(x^2 + \frac{3}{x^2} - \frac{1}{x}\right) dx$

$$= \int_1^3 \left(x^2 + 3x^{-2} - \frac{1}{x}\right) dx$$

$$= \frac{1}{3}x^3 - \frac{3}{x} - \ln|x| \Big|_1^3$$

$$= \left(9 - 1 - \ln 3\right) - \left(\frac{1}{3} - 3 - 0\right) = \frac{32}{3} - \ln 3$$

$$\boxed{\frac{32}{3} - \ln 3 \approx 9.56805}$$

19. Use a definite integral to find the average value of $f(x) = 3 + 6\sin x$ over the interval from $x = 0$ to $x = 2\pi$.

$$\frac{1}{2\pi}\int_0^{2\pi} (3 + 6\sin x)\,dx = \frac{1}{2\pi}\Big[3x - 6\cos x\Big]_{x=0}^{x=2\pi}$$

$$= \frac{1}{2\pi}\Big[(6\pi - 6) - (0 - 6)\Big]$$

$$= \frac{6\pi}{2\pi} = 3$$

$$\boxed{3}$$

20. Use a u-substitution to evaluate the definite integral: $\displaystyle\int_0^2 \frac{x}{\sqrt{1+2x^2}}\,dx$

$u = 1 + 2x^2$

$du = 4x\,dx$

$\frac{1}{4}\,du = x\,dx$

$x = 0 \Rightarrow u = 1$

$x = 2 \Rightarrow u = 9$

$$\int_1^5 \frac{1}{4} u^{-1/2}\,du$$

$$= \frac{1}{2} u^{1/2}\Big|_1^9$$

$$= \frac{3}{2} - \frac{1}{2} = 1$$

$$\boxed{1}$$