Math 131 - Test 2
October 11, 2023

Name Key

Score ________

Show all work to receive full credit. Supply explanations where necessary. Unless otherwise indicated, use differentiation rules for all derivatives and do not simplify.

1. (10 points) Let $f(x) = x^2 - 8x + 5$. Use the **limit definition of the derivative** to determine $f'(x)$. Show all work.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^2 - 8(x+h) + 5] - [x^2 - 8x + 5]}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 8x - 8h + 5 - x^2 + 8x - 5}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 8h}{h} = \lim_{h \to 0} \frac{h(2x + h - 8)}{h}$$

$$= \lim_{h \to 0} (2x + h - 8) = 2x - 8$$

$$\boxed{f'(x) = 2x - 8}$$

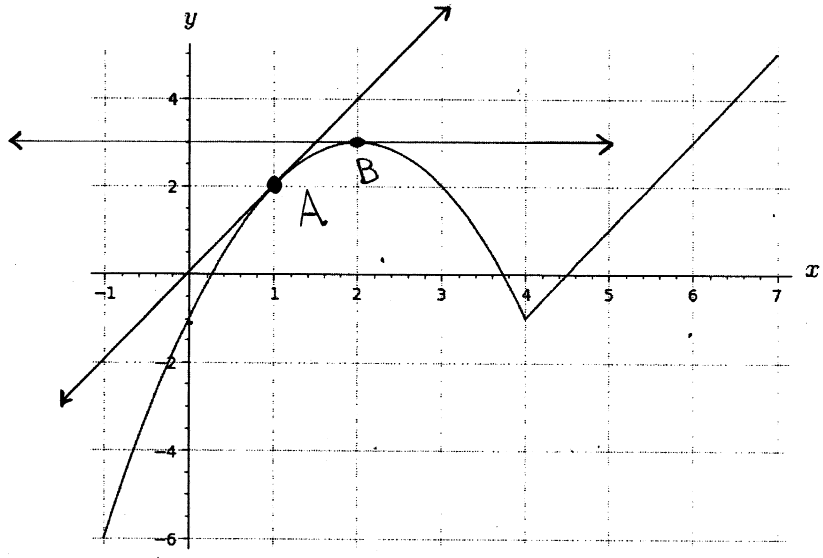
2. (5 points) We studied three specific ways in which a derivative may fail to exist. Describe any two of the three ways. Supply an illustration if it may help.

① f IS NOT DIFFERENTIABLE AT ANY POINT WHERE $\boxed{\text{f IS DISCONTINUOUS.}}$

② f IS NOT DIFFERENTIABLE AT ANY POINT WHERE ITS $\boxed{\text{GRAPH HAS A SHARP POINT.}}$ (TAN. LINE FROM LEFT ≠ TAN LINE FROM RIGHT).

③ f IS NOT DIFFERENTIABLE AT ANY POINT WHERE ITS $\boxed{\text{GRAPH HAS A VERTICAL TANGENT LINE.}}$

3. (10 points) Use the graph of $y = f(x)$ shown below to solve the following problems.



(a) Find a point on the graph at which the derivative exists. Label your point with an A. Then sketch the tangent line through A.

SEE ABOVE

(b) Use your tangent line to estimate the value of the derivative at A.

$\boxed{f'(A) \approx 2}$ THE TANGENT LINE SEEMS TO PASS THROUGH $(1,2)$ AND $(2,4)$.

$\frac{\Delta y}{\Delta x} = \frac{4-2}{2-1} = \frac{2}{1}$

(c) Find another point on the graph at which the derivative exists, but has a different value than above. Label your point with an B. Then sketch the tangent line through B.

SEE ABOVE.

(d) Use your tangent line to estimate the value of the derivative at B.

$\boxed{f'(B) = 0}$ MY TANGENT LINE LOOKS HORIZONTAL.

(e) Find a point on the graph at which the derivative does not exist. Give the x-coordinate of your point, and explain why the derivative does not exist there.

$\boxed{x = 4.}$ THE GRAPH HAS A SHARP POINT SO TANGENT LINE FROM LEFT DOESN'T MATCH TANGENT LINE FROM RIGHT.

4. (8 points) Let $g(x) = x^2(2x+1)^4$. Find the instantaneous rate of change of g at the point where $x = 2$.

$$g'(x) = 2x(2x+1)^4 + x^2(4)(2x+1)^3(2)$$

$$g'(2) = 4(5)^4 + 16(5)^3(2) = \boxed{6500}$$

5. (4 points) Let $F(x) = x^3 - 8x + 10$. Compute the average rate of change of F over the interval $[0, 3]$.

$$\frac{\Delta F}{\Delta x} = \frac{F(3) - F(0)}{3 - 0} = \frac{13 - 10}{3} = \boxed{1}$$

$$F(3) = 27 - 24 + 10 = 13$$

6. (8 points) The following table gives the values of $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ at selected values of x.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-3	4	5	-3	2
1	-3	7	2	-8

(a) Let $h(x) = \sqrt{x} + f(x)g(x)$. Compute $h'(1)$.

$$h'(x) = \frac{1}{2}x^{-1/2} + f'(x)g(x) + f(x)g'(x)$$

$$h'(1) = \frac{1}{2} + f'(1)g(1) + f(1)g'(1) = \frac{1}{2} + (7)(2) + (-3)(-8)$$

$$= 38.5$$

$h(x) =$

(b) Let $\dfrac{x+2}{f(x)}$. Compute $h'(-3)$.

$$h'(x) = \frac{f(x)(1) - (x+2)f'(x)}{(f(x))^2}$$

$$h'(-3) = \frac{f(-3) - (-1)f'(-3)}{[f(-3)]^2} = \frac{4+5}{4^2} = \boxed{\frac{9}{16}}$$

7. (20 points) Determine the derivative of each function. Show all work. Do not simplify.

(a) $y = 6x^5 + \sqrt[5]{x^3} - \dfrac{3}{x^4}$

$$y = 6x^5 + x^{3/5} - 3x^{-4}$$

$$\boxed{\frac{dy}{dx} = 30x^4 + \frac{3}{5}x^{-2/5} + 12x^{-5}}$$

(b) $g(x) = \dfrac{\sin(7x)}{5x - 10}$

$$\boxed{g'(x) = \frac{(5x-10)\cos(7x)(7) - \sin(7x)(5)}{(5x-10)^2}}$$

(c) $f(x) = \tan(x^2 + 1)$

$$f'(x) = \sec^2(x^2+1)(2x) \quad \text{[CHAIN RULE]}$$

OR $\boxed{f'(x) = 2x\sec^2(x^2+1)}$

(d) $y = x^3 \csc x$

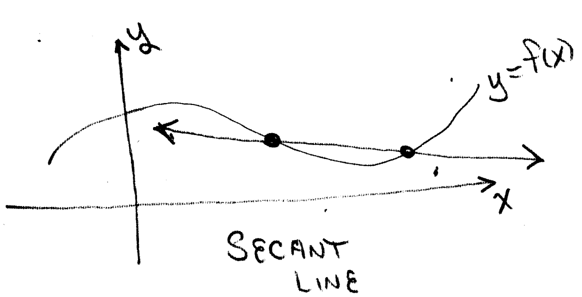
$$\boxed{\frac{dy}{dx} = 3x^2 \csc x - x^3 \csc x \cot x}$$

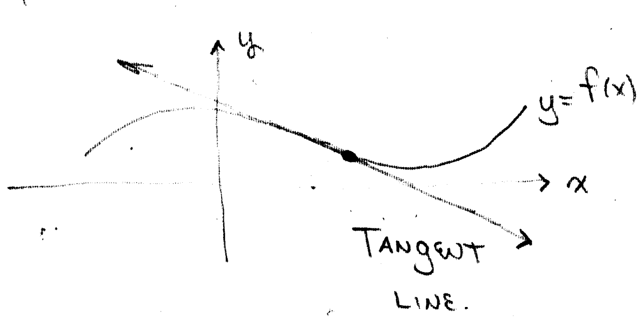
8. (5 points) What is the difference between a secant line and a tangent line? Supply an illustration if it may help.

A SECANT LINE PASSES THROUGH TWO POINTS ON A CURVE.

A TANGENT LINE FOLLOWS FROM SECANT LINES AS THE POINTS GET CLOSER AND CLOSER.

TANGENT LINES SHOW THE SLOPE OF THE CURVE.





9. (6 points) Let $y = \underbrace{3x^4 + 8x}_{\text{10TH DERIV OF THIS PART IS 0.}} - \overbrace{\cos x}^{\text{10TH DERIV OF } -\cos x \text{ IS SAME AS 2ND DERIV}}$. Determine the 10th derivative, $\frac{d^{10}y}{dx^{10}}$.

$\frac{d}{dx}(-\cos x) = \sin x$

$\frac{d}{dx} \sin x = \cos x$

$$\boxed{\frac{d^{10}y}{dx^{10}} = \cos x}$$

10. (10 points) An object is launched vertically upward from over the edge of a building. The object's height (in meters) after t seconds is given by

$$s(t) = -4.9t^2 + 14.7t + 49.$$

Include units with your answer for each part of this problem.

(a) Determine the object's maximum height.

$s'(t) = -9.8t + 14.7$

$s'(t) = 0 \Rightarrow t = \frac{14.7}{9.8} = 1.5 \text{ sec}$

$s(1.5) = -4.9(1.5)^2 + 14.7(1.5) + 49$

$= \boxed{60.025 \text{ m}}$

(b) What is the object's speed when it hits the ground?

$s(t) = 0 \Rightarrow -4.9t^2 + 14.7t + 49 = 0$

$-4.9(t^2 - 3t - 10) = 0$

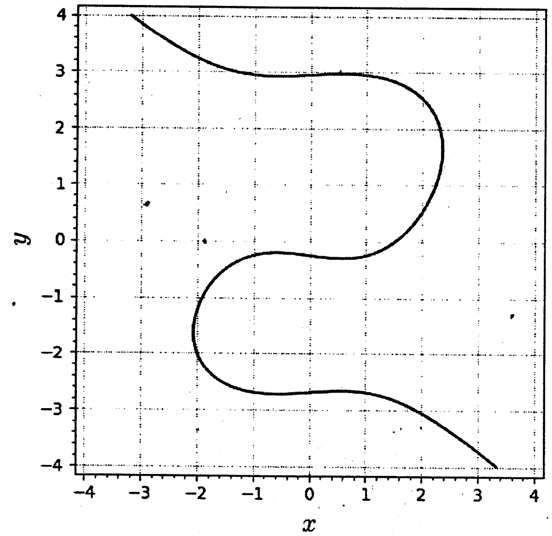
$-4.9(t-5)(t+3) = 0$

$t = 5 \text{ sec}$

$s'(5) = -9.8(5) + 14.7 = -34.3 \text{ m/sec}$

$\text{Speed} = |s'(5)| = \boxed{34.3 \text{ m/sec}}$

11. (14 points) The graph of the equation $x^3 + y^3 = 8y + x + 2$ is shown below.



(a) Use implicit differentiation to find a formula for dy/dx.

$$\frac{d}{dx}\left(x^3+y^3\right) = \frac{d}{dx}\left(8y+x+2\right)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 8\frac{dy}{dx} + 1$$

$$3y^2\frac{dy}{dx} - 8\frac{dy}{dx} = 1-3x^2$$

$$\boxed{\frac{dy}{dx} = \frac{1-3x^2}{3y^2-8}}$$

(b) Use dy/dx to compute the slope of the graph at the point $(-2, -2)$. Then determine an equation of the tangent line at $(-2, -2)$. (If you could not solve part (a), sketch the tangent line and estimate its slope.)

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(-2,-2)} = \frac{1-12}{12-8} = \frac{-11}{4}$$

$$y+2 = -\frac{11}{4}(x+2)$$

or $\quad y = -\frac{11}{4}x - \frac{30}{4}$

(c) Find an equation of the line normal to the graph at the point $(-2, -2)$. (If you could not solve part (b), sketch the normal line and estimate its slope.)

$$m_{\perp} = \frac{4}{11}$$

$$y+2 = \frac{4}{11}(x+2)$$

or $\quad y = \frac{4}{11}x - \frac{14}{11}$