Math 131 - Test 3
November 8, 2023

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) Suppose f and f^{-1} are differentiable functions. The table below shows the values of $f(x)$ and $f'(x)$ at selected values of x. Find $(f^{-1})'(3)$. Show how you got it.

x	0	1	2	3
$f(x)$	5	3	7	2
$f'(x)$	0	8	4	1

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}$$

$$= \boxed{\frac{1}{8}}$$

$f(1) = 3 \Leftrightarrow f^{-1}(3) = 1$

$(f^{-1})(3) = 1$

AND $f'(1) = 8$

2. (7 points) Let $g(x) = (\cos^{-1} x)^2$. Find the **exact value** of $g'(1/2)$. Simplify your answer as much as possible.

$$g'(x) = 2(\cos^{-1} x)\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$g'(1/2) = 2\cos^{-1}\left(\tfrac{1}{2}\right)\left(\frac{-1}{\sqrt{1-1/4}}\right) = 2\left(\frac{\pi}{3}\right)\left(\frac{-1}{\sqrt{\frac{3}{4}}}\right) = \boxed{\frac{-4\pi}{3\sqrt{3}}}$$

3. (4 points) Let $f(x) = 9x + 13$. Find $(f^{-1})'(x)$.

$f'(x) = 9$

So that $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

$$= \boxed{\frac{1}{9}}$$

-OR-

EASY TO ACTUALLY FIND INVERSE...

$f^{-1}(x) = \dfrac{x-13}{9}$

$(f^{-1})'(x) = \dfrac{1}{9}$

4. (4 points) Suppose you know that $a^8 = 4$. Use this to find each of the following. Show how you got your answers.

(a) $\log_a 4 = \boxed{8}$ BECAUSE $a^8 = 4$ OR USE $\ln a^8 = \ln 4$ $\Rightarrow \ln a = \frac{\ln 4}{8}$

(b) $\log_a 2 = \boxed{4}$

$\log_a 2 = \log_a 4^{1/2} = \frac{1}{2}\log_a 4 = \frac{1}{2}(8)$

5. (4 points) Find $h'(x)$ if $h(x) = \log_3[(5x+1)^7]$.

$h(x) = \frac{7}{\ln 3}\ln(5x+1)$

$h'(x) = \frac{7}{\ln 3}\,\frac{5}{5x+1} = \frac{35}{\ln 3\,(5x+1)}$

6. (4 points) Find dy/dx if $y = x^2 e^{\tan x}$.

$\frac{dy}{dx} = 2xe^{\tan x} + x^2 e^{\tan x}\cdot\sec^2 x$

7. (8 points) Use logarithmic differentiation to find dy/dx when $y = (\sin x)^x$.

$\ln y = \ln(\sin x)^x$

$\ln y = x\ln\sin x$

$\frac{1}{y}\frac{dy}{dx} = \ln\sin x + (x)\left(\frac{\cos x}{\sin x}\right) = \ln\sin x + x\cot x$

$\frac{dy}{dx} = (\sin x)^x\cdot\left(\ln\sin x + x\cot x\right)$

8. (8 points) Determine the linearization of $f(x) = x^{1/2} + x^{1/4}$ at $x = 16$. Then use your linearization to approximate $f(16.8)$.

$f(16) = 16^{1/2} + 16^{1/4} = 4 + 2 = 6$

$f'(x) = \frac{1}{2}x^{-1/2} + \frac{1}{4}x^{-3/4}$

$f'(16) = \frac{1}{2}(16)^{-1/2} + \frac{1}{4}(16)^{-3/4}$

$= \frac{1}{8} + \frac{1}{32}$

$= \frac{5}{32}$

$\boxed{L(x) = 6 + \frac{5}{32}(x - 16)}$

$f(16.8) \approx L(16.8)$

$= 6 + \left(\frac{5}{32}\right)(0.8)$

$= 6.125$

9. (6 points) Use differentials to approximate the change in $f(x) = \tan^{-1}(2x)$ as x changes from 0.5 to 0.47.

$f'(x) = \frac{2}{1 + 4x^2}$

$\Delta y \approx \left(\frac{2}{1+4x^2}\right)\Delta x$

$x = 0.5, \quad \Delta x = -0.03$

$\Delta y \approx \left(\frac{2}{1+1}\right)(-0.03) = \boxed{-0.03}$

10. (8 points) Determine the differential dy.

(a) $y = \ln(\cos x)$

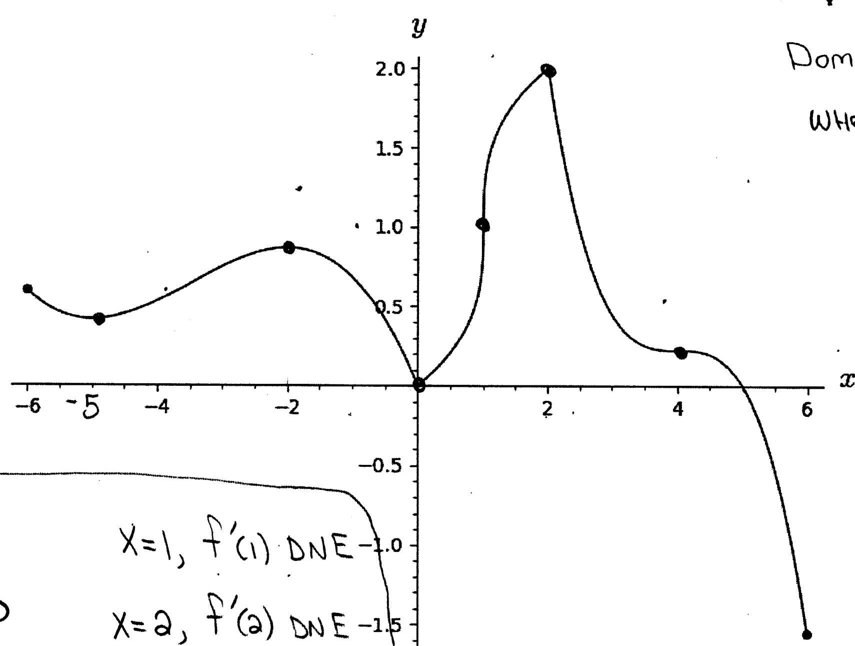
$\frac{dy}{dx} = \frac{1}{\cos x} \cdot (-\sin x) = -\tan x \Rightarrow \boxed{dy = -\tan x \, dx}$

(b) $y = 5^{x^2}$

$\frac{dy}{dx} = 5^{x^2} \ln 5 \cdot 2x \Rightarrow \boxed{dy = 5^{x^2} \ln 5 \cdot 2x \, dx}$

11. (6 points) The graph of $y = f(x)$ is shown below. Estimate the critical numbers of f. Explain your reasoning. (Pay attention to the scale along the x-axis.)

DOMAIN INTERIOR PTS WHERE $f'(x) = 0$ OR $f'(x)$ DNE



$x = -5,\ f'(-5) = 0$ $\qquad$ $x = 1,\ f'(1)$ DNE

$x = -2,\ f'(-2) = 0$ $\qquad$ $x = 2,\ f'(2)$ DNE

$x = 0,\ f'(0)$ DNE $\qquad$ $x = 4,\ f'(4) = 0$

12. (2 points) Referring to the function f is the previous problem, explain why $f(6)$ is not a relative minimum.

RELATIVE EXTREMA OCCUR ONLY AT DOMAIN INTERIOR PTS (BY THEIR DEFINITIONS).

X = 6 IS NOT AN INTERIOR PT IN THE DOMAIN.

13. (8 points) Let $g(x) = x^3 - 3x^2 + 1$ for $0 \leq x \leq 4$. Find the critical numbers of g. Then find the absolute minimum and maximum values of g.

$g'(x) = 3x^2 - 6x$

$g'(x) = 0 \Rightarrow 3x(x-2) = 0$

$\cancel{x = 0}, x = 2$

X = 2 IS THE ONLY CRITICAL NUMBER

END PTS: $x = 0,\ x = 4$

X	g(x)	
2	-3	← ABS MIN.
0	1	
4	17	← ABS MAX

14. (3 points) Let $f(x) = x^2 - 18\ln x$. It is easy to check (don't bother) that

$$f'(x) = \frac{2(x^2-9)}{x}.$$

Looking at f', Steve claimed that $\boxed{x = 3}$, $\cancel{\boxed{x = -3}}$ and $\cancel{\boxed{x \neq 0}}$ are the critical numbers of f. Explain where Steve went wrong.

ALL THOSE NUMBERS MAKE $f'(x) = 0$ OR DNE;

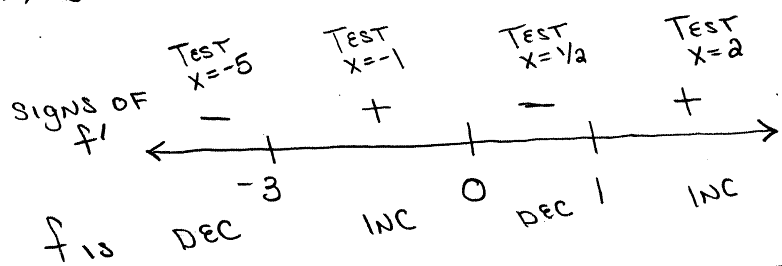
BUT $X = -3$ AND $X = 0$ ARE NOT IN THE DOMAIN OF f.

$X = -3$ AND $X = 0$ ARE NOT CRIT #'S.

15. (7 points) The first derivative of f is given by $f'(x) = x^3(x-1)(x+3)$. Construct a sign chart (or number line) for f' and determine open intervals on which f is increasing/decreasing.

$f'(x) = 0 \Rightarrow X = 0,\ X = 1,\ X = -3$

$f'(x)$ DNE NOWHERE



f IS DECREASING ON $(-\infty,-3) \cup (0,1)$
f IS INCREASING ON $(-3,0) \cup (1,\infty)$

16. (7 points) Find and classify the critical numbers of $F(x) = x^4 - 8x^3 + 18x^2 - 11$.

$F'(x) = 4x^3 - 24x^2 + 36x$

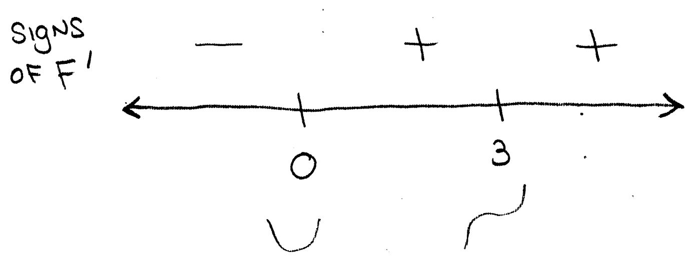
$F'(x) = 0 \Rightarrow 4x(x^2 - 6x + 9) = 0$

$4x(x-3)^2 = 0$

$X = 0,\ X = 3$

$F'(x)$ DNE NOWHERE.

CRIT #'S ARE $X = 0,\ X = 3$



$f(0) = -11$ IS A REL MIN

$f(3)$ IS NEITHER A MIN NOR A MAX.

The following problems are due Monday, November 13, 2023. You must work on your own.

17. (3 points) Suppose that a particle is moving smoothly along the graph $y = e^{-5x}$ in such a way that $\frac{dy}{dt} = 15$ when $x = 0$. Find $\frac{dx}{dt}$ at that point.

$y = e^{-5x}$

$\frac{dy}{dt} = e^{-5x} \cdot -5\frac{dx}{dt}$

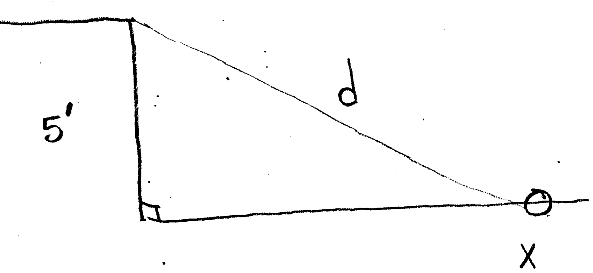
WHEN $x = 0$,

$15 = e^{-5(0)} \cdot -5 \cdot \frac{dx}{dt}$

$15 = (1)(-5)\frac{dx}{dt}$

$\frac{dx}{dt} = -3$

18. (5 points) A fisherman on a dock 5 ft above the surface of the water is slowly reeling in fishing line so that his bobber is approaching the dock (along the water) at a rate of 3.25 ft per minute. Find the rate at which the fishermen is reeling in the fishing line at the moment the bobber is 12 ft from the dock. (This problem is similar to Example 3 in the Lecture 19 notes.)



PYTHAG THEOREM

$25 + x^2 = d^2$

$2x\frac{dx}{dt} = 2d\frac{dd}{dt}$

WHEN $x = 12$, $d^2 = 25 + 144 = 169$
$\Rightarrow d = 13$

$\frac{dx}{dt} = -3.25$

FIND $\frac{dd}{dt}$ WHEN $x = 12$

$2(12)(-3.25) = 2(13)\frac{dd}{dt}$

$\Downarrow$

$\frac{dd}{dt} = \frac{(12)(-3.25)}{13} = -3 \text{ ft/min}$