Math 131 - Final Exam

December 14, 2023

Name key Score

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Use algebraic techniques (not a graph, table, or L'Hôpital's rule) to determine each limit. You may need to use $+\infty$, $-\infty$, or DNE.

   (a) $\lim\limits_{x \to 2} \dfrac{x^3 - x^2 - 2x}{3x - 6}$ 0/0 INDET.

   $$= \lim_{x \to 2} \frac{x\cancel{(x-2)}(x+1)}{3\cancel{(x-2)}} = \frac{2(3)}{3} = 2$$

   (b) $\lim\limits_{x \to 5^-} \dfrac{x^2 + 25}{x - 5}$ 50/0 SOME KIND OF ∞

   LEFT OF 5: $\dfrac{\text{POS}}{\text{NEG}}$

   LIMIT IS $-\infty$

2. (10 points) Each function given below has a single point of discontinuity, and each discontinuity has a specific name. State the x-value at which the function is discontinuous and classify the discontinuity.

   (a) $f(x) = \dfrac{\sin x}{x}$ DISCONT. AT $X=0$. IT IS REMOVABLE

   BECAUSE $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

   (b) $g(x) = \dfrac{(x^2 + 2x)|x - 2|}{x - 2}$ DISCONT. AT $X=2$. JUMP DISCONT

   BECAUSE $\lim\limits_{x \to 2^-} g(x) = -8$ AND $\lim\limits_{x \to 2^+} g(x) = +8$

   (c) $h(x) = \dfrac{x^2 - 16}{x + 3}$ DISCONT. AT $X=-3$. INFINITE DISCONT

   BECAUSE $\dfrac{9-16}{-3+3}$ IS A $\dfrac{\text{NONZERO}}{\text{ZERO}}$ FORM.

3. (10 points) Let $f(x) = 3x^2 + 2x$. Use the **limit definition of the derivative** to determine $f'(x)$. Show all work.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[3(x+h)^2 + 2(x+h)] - [3x^2 + 2x]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 + \cancel{2x} + 2h - \cancel{3x^2} - \cancel{2x}}{h}$$

$$= \lim_{h \to 0} \frac{6xh + 3h^2 + 2h}{h} = \lim_{h \to 0} (6x + 3h + 2)$$

$$= \boxed{6x + 2}$$

4. (10 points) Use basic differentiation rules to determine each derivative. Do not simplify.

(a) $\frac{d}{dt}(e^{4t} \tan^{-1} t) = \boxed{4e^{4t} \tan^{-1} t + \frac{e^{4t}}{1+t^2}}$

(b) $\frac{d}{dx} \sqrt[3]{x^2 + 2x + 1} = \frac{d}{dx}(x^2 + 2x + 1)^{1/3}$

$$= \boxed{\frac{1}{3}(x^2 + 2x + 1)^{-2/3}(2x+2)}$$

5. (10 points) An object is thrown upward in such a way that its height after t seconds is given by $s(t) = -16t^2 + 32t + 128$, where s is measured in feet.

(a) What is the maximum height of the object?

STOPS WHEN

$s'(t) = -32t + 32 = 0$

$\Rightarrow t = 1$

$s(1) = -16 + 32 + 128 = \boxed{144 \text{ FT}}$

(b) What is the object's velocity when it hits the ground?

$s(t) = 0 \Rightarrow -16(t^2 - 2t - 8) = 0$

$-16(t-4)(t+2) = 0$

$t = 4$

$s'(t) = -32t + 32$

$s'(4) = -32(4) + 32$

$= \boxed{-96 \text{ FT/SEC}}$

6. (10 points) Let $f(x) = 7 + 2x^2 - 2x^3$.

(a) Use the 1st derivative test to determine all relative extreme values.

$f'(x) = 4x - 6x^2$

$f'(x) = 0 \Rightarrow 2x(2 - 3x) = 0$

$x = 0, \; x = \frac{2}{3}$

SIGNS OF f': $-$ | $+$ | $-$ at $x = 0$, $x = \frac{2}{3}$

$f(0) = 7$ IS A REL MIN.

$f\left(\frac{2}{3}\right) = \frac{197}{27} \approx 7.3$ IS A REL MAX.

(b) Use the 2nd derivative test to determine open intervals on which the graph of f is concave up/down.

$f''(x) = 4 - 12x$

$f''(x) = 0 \Rightarrow x = \frac{1}{3}$

SIGNS OF f'': $+$ | $-$ at $x = \frac{1}{3}$

GRAPH OF f IS CONCAVE UP ON $(-\infty, \frac{1}{3})$ AND CONCAVE DOWN ON $(\frac{1}{3}, \infty)$.

7. (10 points) Use any analytical method (not a table or graph) to determine each limit.

(a) $\lim_{x\to\infty} \left(\frac{\ln x^4}{x^3}\right)$ ∞/∞

∞/∞

$$= \lim_{x\to\infty} \left(\frac{4\ln x}{x^3}\right) = \lim_{x\to\infty} \frac{4/x}{3x^2} = \lim_{x\to\infty} \frac{4}{3x^3} = 0$$

(b) $\lim_{x\to 0} \left(\frac{x^2 - 2 + 2\cos x}{x^3}\right)$ $0/0$

$$= \lim_{x\to 0} \left(\frac{2x - 2\sin x}{3x^2}\right) \quad 0/0$$

$$= \lim_{x\to 0} \left(\frac{2 - 2\cos x}{6x}\right) \quad 0/0 \quad = \lim_{x\to 0} \frac{2\sin x}{6} = 0$$

8. (10 points) Evaluate each definite integral.

(a) $\int_1^2 \left(e^x + \frac{1}{x}\right)\, dx$

$$= e^x + \ln|x| \Big|_1^2$$

$$= \left[e^2 + \ln(2)\right] - \left[e + \ln(1)\right] = e^2 - e + \ln 2 \approx 5.36$$

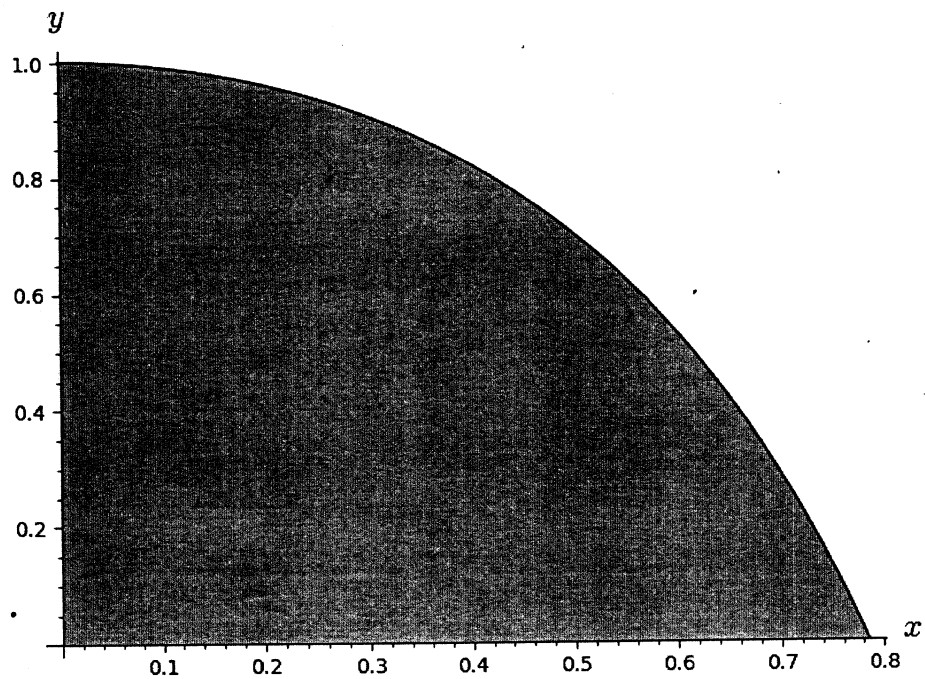
(b) $\int_0^{16} (\sqrt{x} + \sqrt[4]{x})\, dx$

$$= \int_0^{16} \left(x^{1/2} + x^{1/4}\right) dx$$

$$= \frac{x^{3/2}}{3/2} + \frac{x^{5/4}}{5/4} \Big|_0^{16} = \frac{2}{3}x^{3/2} + \frac{4}{5}x^{5/4} \Big|_0^{16}$$

$$= \frac{2}{3}(4^3) + \frac{4}{5}(2^5) = \frac{1024}{15} \approx 68.27$$

9. (10 points) The graph of $f(x) = 2 - \sec^2 x$ on $[0, \pi/4]$ is shown below, and the region under the graph is shaded.



(a) Check that $f(0) = 1$ and $f(\pi/4) = 0$ as the graph indicates.

$f(0) = 2 - \sec^2(0) = 2 - 1 = 1$ ✓

$f(\frac{\pi}{4}) = 2 - \sec^2(\frac{\pi}{4}) = 2 - (\sqrt{2})^2 = 0$ ✓

(b) Use the fundamental theorem of calculus to find the area of the shaded region.

$$\text{Area} = \int_0^{\pi/4} (2 - \sec^2 x)\, dx =$$

$$2x - \tan x \Big|_0^{\pi/4} = \left[2\left(\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right)\right] - [0 - 0]$$

$$= \frac{\pi}{2} - 1 \approx 0.57$$

10. (10 points)

(a) Use an appropriate substitution to evaluate the indefinite integral $\int 6t^2(t^3+1)^4\,dt$.

$$\int 6t^2(t^3+1)^4\,dt$$

$u = t^3+1$

$du = 3t^2\,dt$

$2\,du = 6t^2\,dt$

$$\int 2u^4\,du = \frac{2}{5}u^5 + C$$

$$\boxed{\frac{2}{5}(t^3+1)^5 + C}$$

(b) Use your result from part (a) to determine the value of $\int_0^2 6t^2(t^3+1)^4\,dt$.

$$\int_0^2 6t^2(t^3+1)^4\,dt = \frac{2}{5}(t^3+1)^5\Big|_0^2$$

$$= \frac{2}{5}(9)^5 - \frac{2}{5}(1)^5$$

$$= \frac{2}{5}(9^5-1) = \boxed{\frac{118096}{5} = 23619.2}$$