Math 131 - Test 2
March 10, 2021

Name key

Score ______

Show all work to receive full credit. Supply explanations where necessary. Unless otherwise indicated, use differentiation rules for all derivatives, and do not simplify.

1. (8 points) Let $f(x) = x^2 + x$. Use a limit definition of the derivative to compute $f'(1)$. Show all work.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^2 + (x+h)] - [x^2 + x]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 + \cancel{x} + h - \cancel{x^2} - \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{h(2x + h + 1)}{h} = \lim_{h \to 0} (2x + h + 1) = 2x + 1$$

$$f'(1) = 2(1) + 1 = \boxed{3}$$

2. (4 points) Use your result from above to find an equation of the line tangent to the graph of $f(x) = x^2 + x$ at the point where $x = 1$.

POINT: $x = 1$, $y = f(1) = 2$

$(1, 2)$

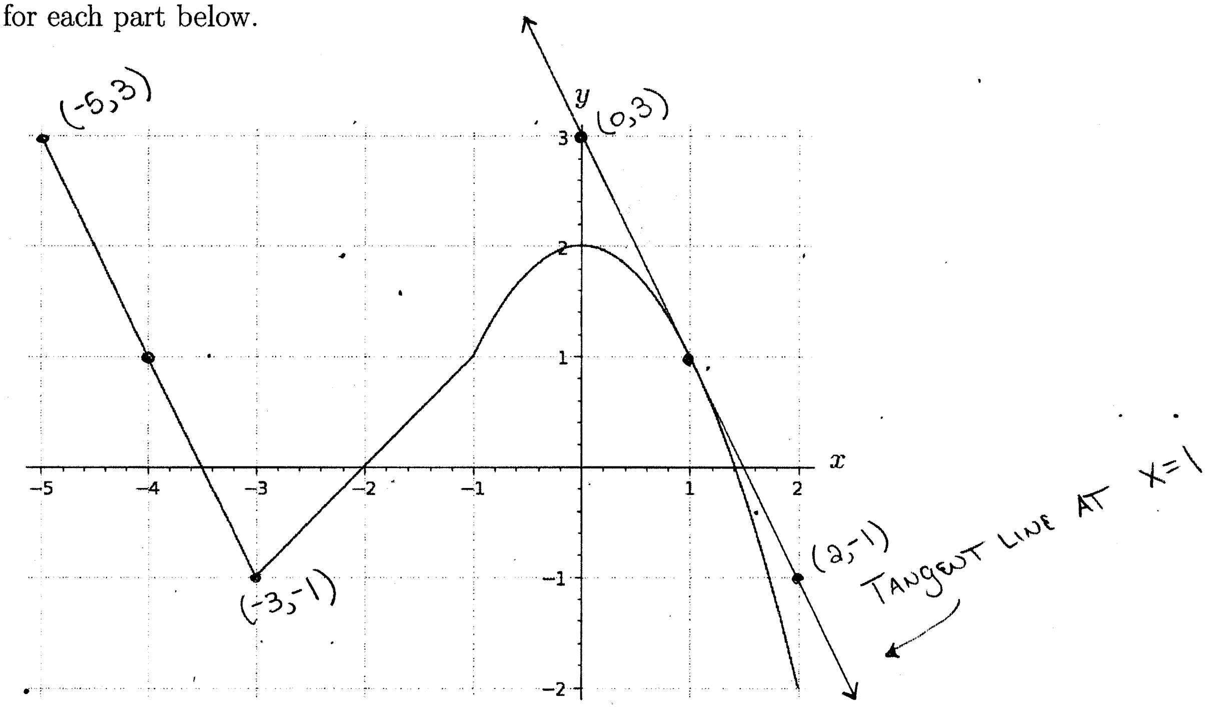
SLOPE: $m = 3$

FROM ABOVE

TANGENT LINE:

$$\boxed{y - 2 = 3(x - 1) \quad \text{OR} \quad y = 3x - 1}$$

3. (12 points) Shown below is the graph of the function g on the interval $[-5, 2]$. Use the graph for each part below.



(a) Identify a point at which $g'(x)$ does not exist and carefully explain how you know.

$\boxed{X=-3}$ THE SHARP POINT ON THE GRAPH INDICATES THAT THERE CANNOT BE A UNIQUE TANGENT LINE AT THAT POINT.

(b) Estimate the value of $g'(1)$. Explain your reasoning or show work.

$\boxed{g'(1)=-2}$ THE SLOPE OF THE TANGENT LINE AT X=1 LOOKS TO BE ABOUT $\frac{-1-3}{2-0} = \frac{-4}{2} = -2.$

(c) Determine the value of $g'(-4)$. Explain your reasoning or show work.

$\boxed{g'(-4)=-2}$ THE SLOPE OF THE TANGENT LINE AT X=-4 IS $\frac{-1-3}{-3-(-5)} = \frac{-4}{2} = -2$

(d) Identify a point at which $g'(x) = 0$ and carefully explain how you know.

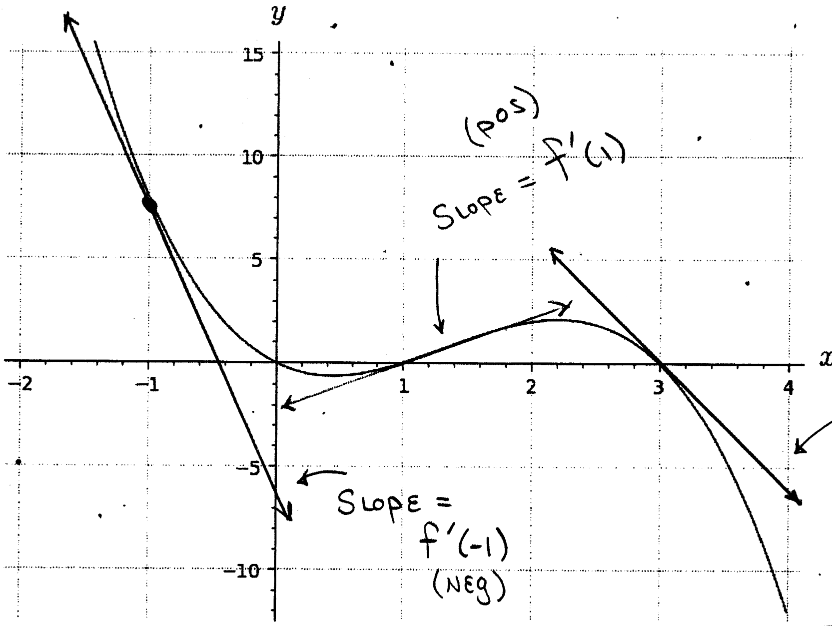
$\boxed{X=0}$ $\boxed{g'(0)=0}$ THE TANGENT LINE AT X=0 IS HORIZONTAL.

4. (6 points) The graph of the function $y = f(x)$ is shown below. Using the graph, place the following values in order from least to greatest. Explain or show work.

$$f'(-1), \qquad f'(1), \qquad f'(3)$$

$f'(1)$ & $f'(3)$ ARE NEGs

BUT THE TAN LINE AT $X=-1$ IS STEEPER $\Rightarrow f'(-1) < f'(3)$.



SLOPE = $f'(3)$ (NEG)

$$\boxed{f'(-1) < f'(3) < f'(1)}$$

5. (6 points) Find an equation of the tangent line to the graph of $y = 2x^3 - 4x^2 - 5x - 3$ at the point where $x = -1$.

$$\frac{dy}{dx} = 6x^2 - 8x - 5$$

$$m = \left.\frac{dy}{dx}\right|_{x=-1} = 6+8-5 = 9$$

POINT: $X = -1$

$$y = -2-4+5-3$$
$$= -4$$

TAN LINE:

$$y+4 = 9(x+1)$$

OR

$$y = 9x+5$$

6. (12 points) The following table gives information about the functions f and g and their derivatives at selected points.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	6	0	-6	-3
1	7	2	-3	-1
2	-7	4	0	1
3	-4	8	1	5

(a) Find $h'(1)$ if $h(x) = xf(x) + 4g(x)$.

$$h'(x) = f(x) + xf'(x) + 4g'(x)$$

$$h'(1) = f(1) + (1)f'(1) + 4g'(1) = 7 + (1)(2) + (4)(-1)$$

$$= \boxed{5}$$

(b) Find $h'(2)$ if $h(x) = f(x)/g(x)$.

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Since $g(2) = 0$, $\boxed{h'(2) \text{ DNE}}$

(c) Find $h'(0)$ if $h(x) = [g(x)]^2$.

$$h'(x) = 2[g(x)]\, g'(x)$$

$$h'(0) = 2g(0)g'(0) = 2(-6)(-3) = \boxed{36}$$

7. (5 points) Use trig identities and the quotient rule to derive our formula for the derivative of $y = \cot x$ from the basic rules for the sine and cosine.

$$y = \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x}$$

$$= \boxed{-\csc^2 x}$$

8. (12 points) A potato is launched vertically upward with an initial velocity of 80 ft/s from a potato gun at the top of an 96-foot-tall building. The distance in feet that the potato travels from the ground after t seconds is given by $s(t) = -16t^2 + 80t + 96$.

(a) Determine when the potato hits the ground.

$$-16t^2 + 80t + 96 = 0$$
$$-16(t^2 - 5t - 6) = 0$$
$$-16(t-6)(t+1) = 0$$
$$\boxed{t = 6 \text{ SECONDS}}$$

(b) Determine the speed of the potato when it hits the ground.

$$S'(t) = -32t + 80$$
$$S'(6) = -112 \text{ FT/SEC} \longrightarrow \boxed{\text{SPEED} = 112 \text{ FT/SEC}}$$

(c) Determine the maximum height of the potato.

$$S'(t) = 0 \Rightarrow -32t + 80 = 0$$
$$\Rightarrow t = \frac{80}{32} = 2.5 \text{ SEC}$$
$$\boxed{S(2.5) = 196 \text{ FT}}$$

9. (5 points) Let $r(x) = x^5 + x^2 - 7\sin x$. Find $r''(x)$.

$$r'(x) = 5x^4 + 2x - 7\cos x$$
$$\boxed{r''(x) = 20x^3 + 2 + 7\sin x}$$

10. (7 points) Let $W(x) = \cos^3(x^2+1)$. Compute $W'(x)$. (Hint: You'll need the chain rule twice.)

$$W(x) = [\cos(x^2+1)]^3$$
$$W'(x) = 3[\cos(x^2+1)]^2 \frac{d}{dx} \cos(x^2+1)$$
$$= 3\cos^2(x^2+1)(-\sin(x^2+1))(2x)$$
$$= \boxed{-6x\cos^2(x^2+1)\sin(x^2+1)}$$

11. (15 points) Differentiate. Do not simplify.

(a) $\frac{d}{dx}(x + \csc x)(1 - \tan x)$

PRODUCT RULE...

$$= (1 - \csc x \cot x)(1 - \tan x) + (x - \csc x)(-\sec^2 x)$$

(b) $\frac{d}{d\theta}\sec(\theta^2)$ $= \sec(\theta^2)\tan(\theta^2)\,(2\theta)$

CHAIN RULE...

(c) $\frac{d}{dt}\sqrt{t^3 + 2t + 1}$ $= \frac{d}{dt}\left(t^3 + 2t + 1\right)^{1/2}$

$$= \frac{1}{2}\left(t^3 + 2t + 1\right)^{-1/2}(3t^2 + 2) = \frac{3t^2 + 2}{2\sqrt{t^3 + 2t + 1}}$$

12. (8 points) Assume that y is implicitly defined as a function of x by the equation $x^4 y + y^3 = -2$. Use implicit differentiation to find dy/dx at $(-1, -1)$.

$$\frac{d}{dx}\left(x^4 y + y^3\right) = \frac{d}{dx}(-2)$$

$$4x^3 y + x^4 \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0$$

$$x^4 \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = -4x^3 y$$

$$(x^4 + 3y^2)\frac{dy}{dx} = -4x^3 y$$

$$\frac{dy}{dx} = \frac{-4x^3 y}{x^4 + 3y^2}$$

$$\left.\frac{dy}{dx}\right|_{(-1,-1)} = \frac{-4(-1)(-1)}{1 + 3(1)} = \frac{-4}{4} = \boxed{-1}$$