Math 131 - Final Exam

May 12, 2021

Name key

Score

Show all work to receive full credit. For each problem, place your final answer in the box provided. Each problem is worth 5 points–up to 2 points for the answer and up to 3 points for the supporting work or explanation.

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1. Determine the limit. Use algebraic techniques (not a graph, table, or L'Hôpital's rule) to show how you got your answer.

0/0 MORE WORK!

$$\lim_{x\to 4}\left(\frac{\sqrt{x+5}-3}{x-4}\right)$$

$$\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}\cdot\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}=\lim_{x\to 4}\frac{\cancel{x+5-9}}{\cancel{(x-4)}(\sqrt{x+5}+3)}$$

$$=\lim_{x\to 4}\frac{1}{\sqrt{x+5}+3}=\frac{1}{6}$$

$$\boxed{1/6}$$

2. Determine the limit. Show analytically (not with a graph or table) how you got your answer.

0/0 MORE WORK.

$$\lim_{x\to 1^-}\left(\frac{2x-2}{x^2-2x+1}\right)$$

2/0 SOME KIND OF INF. LIMIT.

$$\lim_{x\to 1^-}\frac{2(x-1)}{(x-1)^2}=\lim_{x\to 1^-}\frac{2}{x-1}$$

FROM THE LEFT OF 1, X−1 IS NEGATIVE

$\Rightarrow$ LIMIT IS $-\infty$

$$\boxed{-\infty}$$

3. Yes or No: Is g continuous at $x = 0$? Use the definition of continuity to support your answer.

$$g(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1 = g(0)$$

Yes!

$$\boxed{\text{Yes.} \quad \lim_{x \to 0} g(x) = g(0)}$$

4. Let $f(x) = x - x^2$. Write $f'(x)$ in the box, then use the limit definition of derivative to obtain your answer.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left[(x+h) - (x+h)^2\right] - \left[x - x^2\right]}{h}$$

$$= \lim_{h \to 0} \frac{x + h - x^2 - 2xh - h^2 - x + x^2}{h} = \lim_{h \to 0} \frac{h - 2xh - h^2}{h}$$

$$= \lim_{h \to 0} 1 - 2x - h = 1 - 2x$$

$$\boxed{f'(x) = 1 - 2x}$$

5. Let $f(x) = \dfrac{\tan x}{2e^x}$. Compute $f'(0)$.

QUOTIENT RULE

$$f'(x) = \frac{2e^x(\sec^2 x) - (\tan x)(2e^x)}{(2e^x)^2}$$

$$f'(0) = \frac{2-0}{4} = \frac{1}{2}$$

$$\boxed{1/2}$$

6. A ball is thrown straight upward in such a way that its height (in feet) after t seconds is given by

$$h(t) = -16t^2 + 48t + 144.$$

How high is the ball one second after it reaches its maximum height?

MAX HEIGHT WHEN

$$h'(t) = -32t + 48 = 0$$

$$t = \frac{48}{32} = \frac{3}{2}$$

$$h\left(\tfrac{3}{2}+1\right) = h(2.5) = 164$$

$$\boxed{164 \text{ FT}}$$

7. Find $\dfrac{dy}{dx}$ if $y = (x^2 + \tan^{-1} x)^3$.

CHAIN RULE

$$\frac{dy}{dx} = 3\left(x^2 + \tan^{-1} x\right)^2 \left(2x + \frac{1}{1+x^2}\right)$$

$$\boxed{\frac{dy}{dx} = 3\left(x^2 + \tan^{-1} x\right)^2 \left(2x + \frac{1}{1+x^2}\right)}$$

8. Find an equation of the line normal to the graph of $x^3 + y^2 = xy + 3$ at the point $(1, 2)$.

$$3x^2 + 2y\frac{dy}{dx} = y + x\frac{dy}{dx}$$

$$3x^2 - y = (x - 2y)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3x^2 - y}{x - 2y}$$

$$m = \left.\frac{dy}{dx}\right|_{(1,2)} = \frac{1}{-3}$$

$$m_\perp = 3$$

$$\boxed{y - 2 = 3(x-1) \quad \text{-or-} \quad y = 3x - 1}$$

9. Let $g(x) = \dfrac{(x+1)^2}{(x+3)^4}$. Use logarithmic differentiation to find $g'(x)$.

$$\ln g(x) = \ln\left(\frac{(x+1)^2}{(x+3)^4}\right) = 2\ln(x+1) - 4\ln(x+3)$$

$$\frac{g'(x)}{g(x)} = \frac{2}{x+1} - \frac{4}{x+3}$$

$$\boxed{g'(x) = \frac{(x+1)^2}{(x+3)^4}\left[\frac{2}{x+1} - \frac{4}{x+3}\right]}$$

10. Find the linearization of $f(x) = \ln x$ at $x = 1$, and use it to approximate $\ln 1.05$.

$$f(1) = 0$$

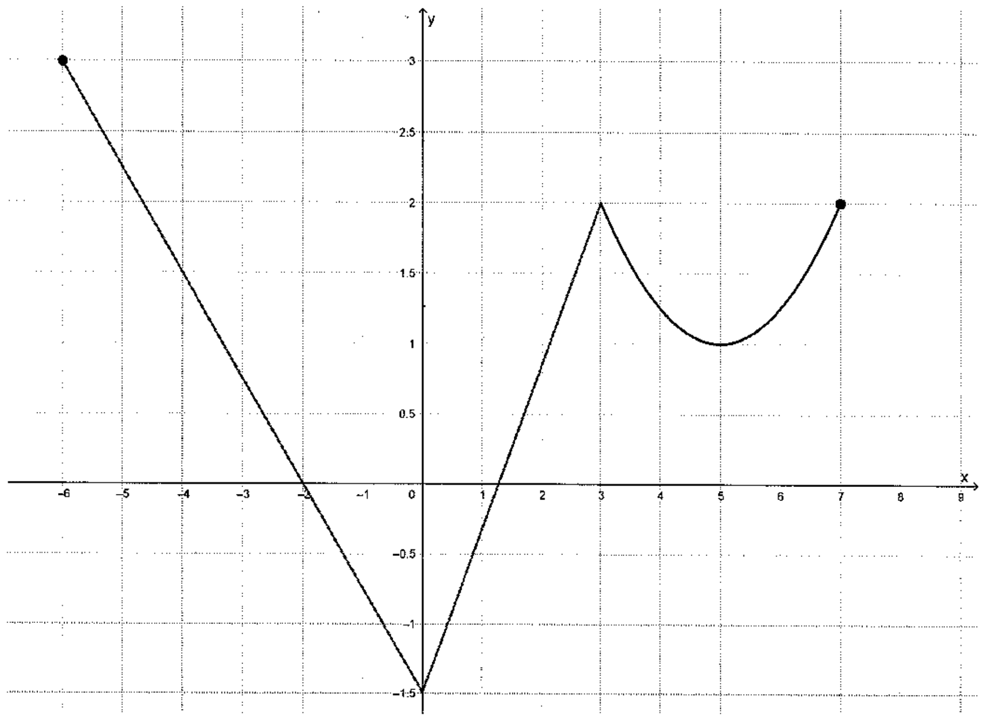
$$f'(x) = \frac{1}{x}, \quad f'(1) = 1$$

$$\Rightarrow \quad L(x) = 0 + 1(x-1)$$

$$L(x) = x - 1$$

$$\boxed{L(x) = x - 1, \quad \ln 1.05 \approx L(1.05) = 0.05}$$

11. The graph of $y = f(x)$ is shown below. Find each critical point of f and say why it is a critical point.



$$\boxed{\begin{array}{ll} x = 0 & \text{BECAUSE } f'(0) \text{ DNE} \\ x = 3 & \text{BECAUSE } f'(3) \text{ DNE} \\ x = 5 & \text{BECAUSE } f'(5) = 0 \end{array}}$$

12. Use calculus techniques to find the absolute extreme values of $f(x) = x^3 - 6x^2 - 6$ on $[-1, 7]$.

$f'(x) = 3x^2 - 12x = 3x(x-4) = 0$

$x = 0,\ x = 4$ (CRIT PTS)

$x = -1,\ x = 7$ (END PTS)

$f(0) = -6$

$f(4) = -38 \leftarrow$ ABS MIN

$f(-1) = -13$

$f(7) = 43 \leftarrow$ ABS MAX

$$\boxed{f(4) = -38 \text{ ABS MIN},\quad f(7) = 43 \text{ ABS MAX}}$$

13. The first derivative of f is given by $f'(x) = x^3(x-1)(x+3)$. Locate the relative extreme values of f.

$f'(x) = 0 \Rightarrow x = 0, 1, -3$

Signs of f': $-$ | $+$ | $-$ | $+$

-3 0 1

f is: DEC | INC | DEC | INC

$f(-3)$ IS A REL MIN.

$f(0)$ IS A REL MAX.

$f(1)$ IS A REL. MIN

14. Find the inflection point(s) of the graph of $f(x) = x^3 - 6x^2 - 6$.

$f'(x) = 3x^2 - 12x$

$f''(x) = 6x - 12 = 0$

$\Rightarrow x = 2$

Signs of f'': $-$ | $+$

Graph is: CD | $x=2$ | CU

CHANGES CONCAVITY

$(2, -22)$ IS THE ONLY INF. POINT.

15. Evaluate the limit: $\lim_{x \to 1} \dfrac{\ln x^3}{x^2 - 1}$

0/0 More work

$$\underbrace{\lim_{x \to 1} \frac{\ln x^3}{x^2 - 1}}_{\text{L'Hôpital's rule}} = \lim_{x \to 1} \frac{\frac{3x^2}{x^3}}{2x} = \frac{\frac{3}{1}}{2} = \boxed{\frac{3}{2}}$$

$3/2$

16. Find $f(x)$ if $f'(x) = 6x^2 + e^x - \sin x$ and $f(0) = 5$.

$$f(x) = \int \left(6x^2 + e^x - \sin x\right) dx$$

$$= 2x^3 + e^x + \cos x + C$$

$$f(0) = 5 \Rightarrow 2(0)^3 + e^0 + \cos 0 + C = 5$$

$$2 + C = 5 \Rightarrow C = 3$$

$$\boxed{f(x) = 2x^3 + e^x + \cos x + 3}$$

17. Let $f(x) = \dfrac{1}{x}$. Use 4 subintervals of equal length and right endpoints of the subintervals to compute a Riemann sum for f on $[1, 2]$.

$$\Delta x = \frac{1}{4} = 0.25$$

$$x_0 = 1, \quad \underset{\uparrow \atop c_1}{x_1 = 1.25}, \quad \underset{\uparrow \atop c_2}{x_2 = 1.5}, \quad \underset{\uparrow \atop c_3}{x_3 = 1.75}, \quad \underset{\uparrow \atop c_4}{x_4 = 2}$$

$$\text{Riemann sum} = \frac{1}{1.25}(0.25) + \frac{1}{1.5}(0.25) + \frac{1}{1.75}(0.25) + \frac{1}{2}(0.25)$$

$$\approx 0.6345$$

$$\boxed{0.6345}$$

18. Evaluate the definite integral.

$$\int_1^2 \left(\frac{1}{x} + 2 + x^3\right)\, dx$$

$$= \ln|x| + 2x + \frac{1}{4}x^4 \Big|_1^2$$

$$= \left(\ln 2 + 4 + 4\right) - \left(\ln 1 + 2 + \frac{1}{4}\right) =$$

$$\ln 2 + \frac{23}{4}$$

$$\boxed{\ln 2 + \frac{23}{4} \approx 6.4431}$$

19. Find the area of the region between the graph of $y = \sqrt{x}$ and the x-axis over the interval $[0, 4]$.

$$\int_0^4 \sqrt{x}\, dx = \int_0^4 x^{1/2}\, dx = \frac{2}{3}x^{3/2}\Big|_0^4$$

$$= \frac{2}{3}4^{3/2} = \frac{16}{3}$$

$$\boxed{\frac{16}{3}}$$

20. In order to evaluate the following integral, an appropriate u-substitution should be made. Carry out the substitution and write the new integral. DO NOT EVALUATE the new integral.

$$\int 5xe^{-x^2}\, dx = -\frac{1}{2}\int 5e^u\, du$$

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

$$\boxed{u = -x^2 \Rightarrow -\frac{5}{2}\int e^u\, du}$$