Math 131 - Test 2
March 9, 2022

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. Unless otherwise indicated, use differentiation rules for all derivatives, and do not simplify.

1. (10 points) Let $f(x) = 7 + 5x - x^2$. Use a limit definition of the derivative to determine $f'(x)$. Show all work.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[7+5(x+h)-(x+h)^2] - [7+5x-x^2]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{7} + \cancel{5x} + 5h - \cancel{x^2} - 2xh - h^2 - \cancel{7} - \cancel{5x} + \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{5h - 2xh - h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(5-2x-h)}{\cancel{h}}$$

$$= \lim_{h\to 0} (5-2x-h) = \boxed{5-2x}$$

2. (5 points) Use differentiation rules to confirm your derivative above. Then find an equation of the line tangent to the graph of $f(x) = 7 + 5x - x^2$ at the point where $x = -1$.

$$f'(x) = \frac{d}{dx}(5x+7) - \frac{d}{dx} x^2$$

$$= 5 - 2x \quad \checkmark$$

$m = f'(-1) = 7$

Point: $x = -1$, $y = f(-1) = 1$

$(-1, 1)$

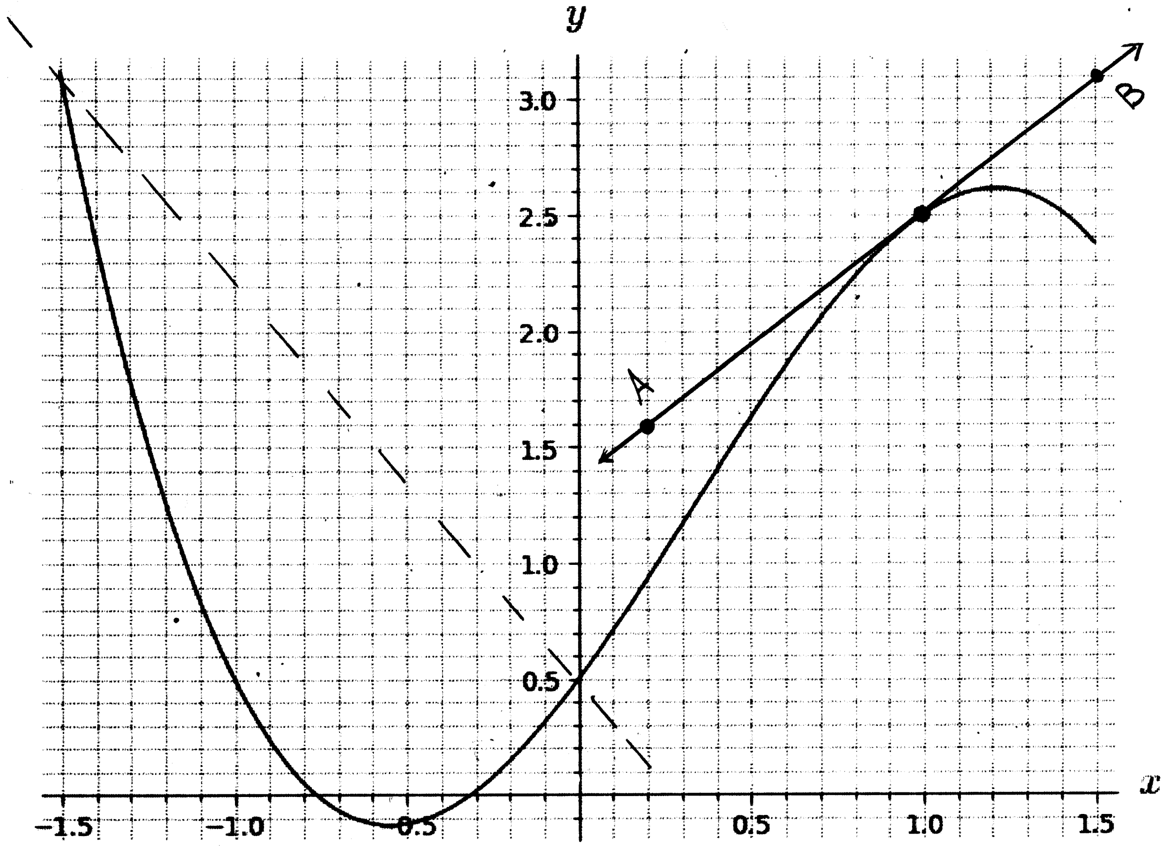
Tan line:

$y - 1 = 7(x+1)$

or

$y = 7x + 8$

3. (9 points) The graph of $y = f(x)$ is shown below. Use the graph for each part of this problem.



(a) Sketch the tangent line at $x = 1$. Then use your tangent line to estimate $f'(1)$. Show work or explain your reasoning.

SEE ABOVE. TO ESTIMATE $f'(1)$, I'LL USE THE POINTS $A(0.2, 1.6)$ AND $B(1.5, 3.1)$, WHICH ARE CLOSE TO THE TANGENT LINE:

$$f'(1) \approx \frac{3.1-1.6}{1.5-0.2} \approx \boxed{1.15}$$

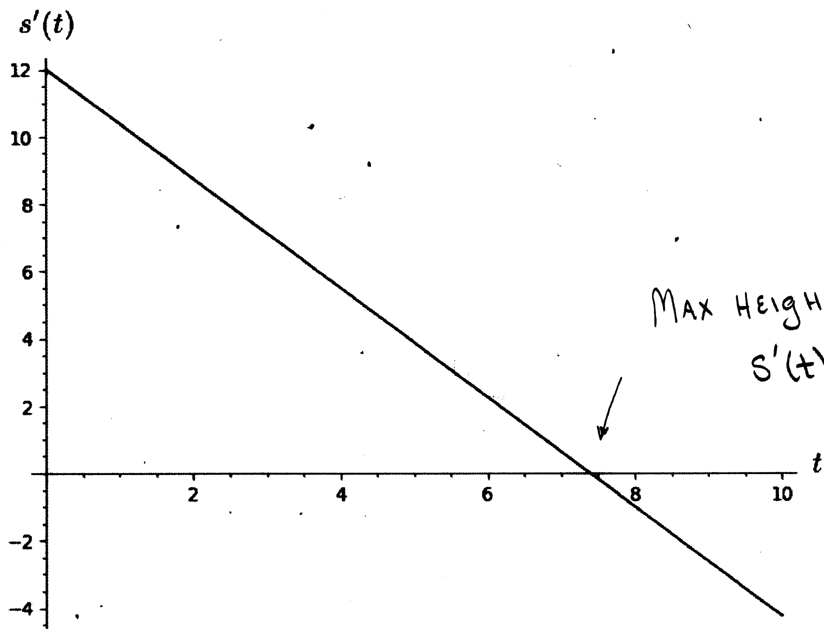
(b) Estimate the interval(s) on which $f'(x) > 0$. Briefly explain your reasoning.

$f'(x) > 0$ WHERE GRAPH MOVES UPHILL. $\boxed{\text{Approx } (-0.55, 1.2)}$

(c) Use a secant line to estimate the average rate of change of f over the interval from $x = -1.5$ to $x = 0$.

$$\frac{\Delta f}{\Delta x} = \frac{f(0) - f(-1.5)}{0 - (-1.5)} \approx \frac{0.5 - 3.1}{1.5} \approx \boxed{-1.73}$$

4. (4 points) Suppose that an object is thrown straight upward and its height after t seconds is given by the function $s(t)$. The graph of the derivative, $s'(t)$, is shown below. Use the graph to estimate when the object reaches its maximum height. Briefly explain your reasoning.



MAX HEIGHT WHEN $s'(t) = 0$. THE OBJECT MOMENTARILY STOPS MOVING UPWARD, THEN BEGINS TO FALL.

$s'(t) = 0$ WHEN $\boxed{t \approx 7.4}$

5. (3 points) Refer to the problem above for which the graph of $s'(t)$ is shown. Recall that $s''(t)$ is the acceleration function for the object. Use the graph to estimate the acceleration at $t = 4$. Show work or explain.

$S'(t)$ IS A LINEAR FUNCTION.

$S''(t)$ = SLOPE OF THE GRAPH OF $S'(t)$ = SOME CONSTANT

USING $(0, 12)$ AND $(7.4, 0)$,

$$S''(t) \approx \frac{0-12}{7.4-0} \approx \boxed{-1.6}$$

6. (20 points) Determine the derivative of each function. Show all work. Do not simplify.

(a) $y = 6x^5 + 7x - 9 + x^\pi - \frac{3}{x^4}$ $= 6x^5 - 7x - 9 + x^\pi - 3x^{-4}$

$$\boxed{\frac{dy}{dx} = 30x^4 + 7 + \pi x^{\pi - 1} + 12x^{-5}}$$

(b) $g(x) = \frac{x^2}{x^3 + 4}$

$$g'(x) = \frac{(x^3+4)(2x) - (x^2)(3x^2)}{(x^3+4)^2} = \boxed{\frac{8x - x^4}{(x^3+4)^2}}$$

(c) $f(t) = \tan(\sqrt{t})$

$$f'(t) = \sec^2(\sqrt{t})\ \frac{d}{dt}\sqrt{t}$$

$$= \sec^2(\sqrt{t})\left(\frac{1}{2}t^{-1/2}\right) = \boxed{\frac{\sec^2(\sqrt{t})}{2\sqrt{t}}}$$

(d) $y = (5x+2)^4(2x+7)^6$

$$\boxed{\frac{dy}{dx} = 4(5x+2)^3(5)(2x+7)^6 + (5x+2)^4(6)(2x+7)^5(2)}$$

7. (6 points) Let $G(x) = x^6 \sin x$. Find $G''(x)$.

$$G'(x) = 6x^5 \sin x + x^6 \cos x$$

$$G''(x) = 30x^4 \sin x + 6x^5 \cos x + 6x^5 \cos x - x^6 \sin x$$

$$= \boxed{(30x^4 - x^6)\sin x + 12x^5 \cos x}$$

8. An object is launched vertically upward from over the edge of a building. The object's height (in meters) after t seconds is given by

$$s(t) = -4.9t^2 + 14.7t + 49.$$

Include units with your answer for each part of this problem.

(a) (3 points) Determine the average rate of change the object's height over the interval from $t = 0$ to $t = 3$.

$$\frac{\Delta s}{\Delta t} = \frac{s(3) - s(0)}{3 - 0} = \frac{49 - 49}{3} = \frac{0}{3} = \boxed{0 \text{ m/s}}$$

(b) (3 points) Determine the object's velocity at time $t = 4$.

$$s'(t) = -9.8t + 14.7$$

$$s'(4) = \boxed{-24.5 \text{ m/s}}$$

(c) (2 points) What is the acceleration of the object?

$$s''(t) = \boxed{-9.8 \text{ m/s}^2}$$

(d) (4 points) Determine the object's maximum height.

$$s'(t) = 0 \Rightarrow t = \frac{14.7}{9.8} = 1.5 \qquad s(1.5) = \boxed{60.025 \text{ m}}$$

(e) (3 points) When does the object hit the ground?

$$s(t) = 0 \Rightarrow -4.9(t^2 - 3t - 10) = 0$$

$$-4.9(t-5)(t+2) = 0 \Rightarrow t = \boxed{5 \text{ s}}$$

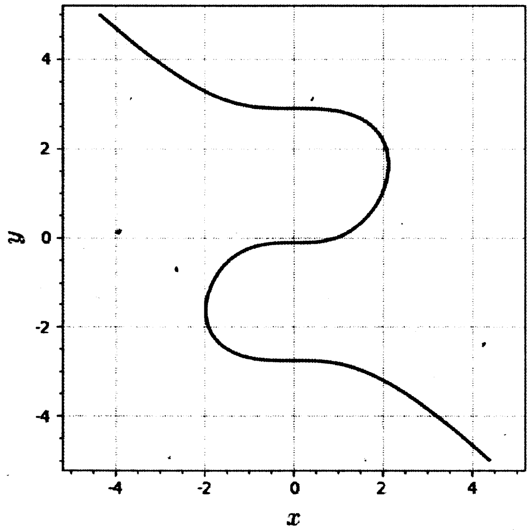
(f) (1 point) What is the object's initial speed?

$$s'(0) = \boxed{14.7 \text{ m/s}}$$

(g) (1 point) What is the object's speed when it hits the ground?

$$s'(5) = -34.3 \text{ m/s} \Rightarrow \text{Speed} = \boxed{34.3 \text{ m/s}}$$

9. (12 points) The graph of the equation $x^3 + y^3 = 8y + 1$ is shown below.



(a) Use implicit differentiation to find a formula for dy/dx.

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(8y+1)$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 8\frac{dy}{dx}$$

$$3x^2 = (8-3y^2)\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{3x^2}{8-3y^2}}$$

(b) Use dy/dx to compute the slope of the graph at the point $(2, 1)$. Then determine an equation of the tangent line at $(2, 1)$.

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = \frac{12}{8-3} = \frac{12}{5}$$

TAN LINE:

$$y - 1 = \frac{12}{5}(x-2) \quad \text{OR} \quad y = \frac{12}{5}x - \frac{19}{5}$$

10. (6 points) Suppose the function f is defined for all x. Describe three ways in which $f'(x)$ may fail to exist.

① IF f IS DISCONTINUOUS AT A POINT, IT CANNOT BE DIFFERENTIABLE THERE.

② IF f IS CONTINUOUS AT A POINT BUT ITS GRAPH HAS A SHARP POINT, THEN $f'(x)$ CANNOT EXIST THERE.

③ IF THE GRAPH OF f HAS A VERTICAL TANGENT LINE AT A POINT, THEN $f'(x)$ CANNOT EXIST THERE.

11. (8 points) The following table gives the values of $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ at selected values of x.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	−1	3	−5
2	1	0	−1	−2

(a) Let $h(x) = \dfrac{1}{x^2} + \dfrac{g(x)}{f(x)}$. Compute $h'(1)$.

$$h'(x) = \frac{-2}{x^3} + \frac{f(x)g'(x) - f'(x)g(x)}{f(x)^2}$$

$$h'(1) = -2 + \frac{(3)(-5) - (-1)(3)}{3^2} = -2 + \frac{-15+3}{9} = -2 - \frac{12}{9}$$

$$\boxed{-\frac{10}{3}}$$

(b) Let $h(x) = g(f(x))$. Compute $h'(2)$.

$$h'(x) = g'(f(x))\, f'(x)$$

$$h'(2) = g'(f(2))\, f'(2) = g'(1)(0) = (-5)(0) = \boxed{0}$$