Math 131 - Test 3
April 13, 2022

Name key

Score ____________

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) The function $f(x) = 2x^3 + x - 3$ has an inverse function. Call it g. Find $g'(0)$ and $g'(-3)$.

$g(x) = f^{-1}(x)$

$g'(x) = \dfrac{1}{f'(f^{-1}(x))}$

$f'(x) = 6x^2 + 1$

$x = 0.$

$f^{-1}(0) = \omega$

$2\omega^3 + \omega - 3 = 0$

$\omega = 1$

$\boxed{g'(0) = \dfrac{1}{f'(1)} = \dfrac{1}{7}}$

$x = -3$

$f^{-1}(-3) = \omega$

$2\omega^3 + \omega - 3 = 0$

$\omega = 0$

$\boxed{g'(-3) = \dfrac{1}{f'(0)} = 1}$

2. (6 points) Compute the slope of the line tangent to the graph of $y = x\sin^{-1}(2x)$ at the point where $x = 1/4$. Write your answer in exact form, simplified as much as possible.

$$\frac{dy}{dx} = \sin^{-1}(2x) + \frac{2x}{\sqrt{1-4x^2}}$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{1}{4}} = \sin^{-1}\left(\frac{1}{2}\right) + \frac{1/2}{\sqrt{1-\frac{1}{4}}} = \boxed{\frac{\pi}{6} + \frac{1}{\sqrt{3}}}$$

3. (6 points) Let $g(x) = e^{-x^2}$. Find $g''(x)$.

$$g'(x) = e^{-x^2}(-2x) = -2xe^{-x^2}$$

$$g''(x) = -2e^{-x^2} - 2xe^{-x^2}(-2x)$$

$$\boxed{g''(x) = -2e^{-x^2} + 4x^2e^{-x^2}}$$

4. (6 points) Let $f(x) = \log_3 [(8x^2 + x)^4]$. Compute $f'(1)$. Write your final answer in decimal form rounded to the nearest thousandth.

$$f(x) = \frac{4}{\ln 3} \ln (8x^2 + x)$$

$$f'(1) = \frac{4}{\ln 3} \cdot \frac{17}{9} = \frac{68}{9 \ln 3}$$

$$f'(x) = \frac{4}{\ln 3} \frac{16x + 1}{8x^2 + X}$$

$$\boxed{f'(1) \approx 6.877}$$

5. (8 points) Use logarithmic differentiation to find $\frac{dy}{dx}$ when $y = \frac{x \cos x}{e^{5x}(x^2 + 1)^3}$.

$$\ln y = \ln x + \ln \cos x - \ln e^{5x} - \ln (x^2+1)^3$$

$$= \ln x + \ln \cos x - 5x - 3 \ln (x^2+1)$$

DERIVATIVE ...

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{X} - \frac{\sin X}{\cos X} - 5 - \frac{6x}{X^2+1} \Rightarrow \boxed{\frac{dy}{dx} = y\left(\frac{1}{X} - \tan X - 5 - \frac{6x}{X^2+1}\right)}$$

6. (6 points) Some values of $f(x)$ and $f'(x)$ near $x = 2$ are given in the table below.

x	1.50	1.75	2.00	2.25	2.50
$f(x)$	6.08	6.90	8.00	9.41	11.14
$f'(x)$	2.74	3.82	5.00	6.26	7.60

Determine the linearization of f at $x = 1.75$, and use it to approximate $f(1.81)$.

$$L(x) = f(1.75) + f'(1.75)(x - 1.75)$$

$$\boxed{L(x) = 6.90 + 3.82(x - 1.75)}$$

$$f(1.81) \approx L(1.81) = 6.90 + 3.82(0.06) = 7.1292$$

$$\boxed{f(1.81) \approx 7.13}$$

7. (10 points) Determine the differential dy.

(a) $y = 5^{x^2+1}$

$$\frac{dy}{dx} = 5^{x^2+1}(2x)\ln 5$$

$$\boxed{dy = 5^{x^2+1}(2x\ln 5)\,dx}$$

(b) $y = \cot^{-1}(\sqrt{x})$

$$\frac{dy}{dx} = \frac{-1}{1+(\sqrt{x})^2}\cdot\frac{d}{dx}\sqrt{x} = \frac{-1}{1+x}\cdot\frac{1}{2\sqrt{x}}$$

$$\boxed{dy = \frac{-1}{2\sqrt{x}(1+x)}\,dx}$$

8. (6 points) Use differentials to approximate the change in $y = \dfrac{1}{1-x}$ as x changes from 2 to 1.98.

$$\frac{dy}{dx} = \frac{1}{(1-x)^2}$$

$$\Delta y \approx \frac{1}{(1-x)^2}\Delta x$$

$x = 2$, $\Delta x = -0.02$

$$\Delta y \approx \frac{1}{(1-2)^2}(-0.02)$$

$$= \boxed{-0.02}$$

9. (6 points) Suppose that the percent error in measuring the side length of a cube is 2%. Use differentials to estimate the percent error in computing the cube's volume.

$V = x^3$

$\frac{dV}{dx} = 3x^2$

$\Delta V \approx 3x^2\,\Delta x$

$\Delta x = 2\%$ OF x

$= 0.02x$

$\Downarrow$

$\Delta V = 3x^2(0.02x) = 0.06x^3$

$\frac{\Delta V}{V} = \frac{0.06x^3}{x^3} = 0.06$

$\Downarrow$

$\boxed{6\%}$

10. (6 points) Find the critical numbers of $g(x) = \dfrac{x^2}{x-1}$. $, x \neq 1$

DOMAIN OF $g = (-\infty, 1) \cup (1, \infty)$

$$g'(x) = \frac{2x(x-1) - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2}$$

$g'(x)$ DNE $\Rightarrow x = 1$

BUT $x = 1$ IS NOT IN THE DOMAIN.

$g'(x) = 0 \Rightarrow x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0,\ x = 2$

CRIT #'S ARE $x = 0,\ x = 2$.

11. (8 points) Use calculus techniques to find the absolute minimum and maximum values of $f(x) = x^4 - 2x^2 + 1$ on $[0, 2]$.

$f'(x) = 4x^3 - 4x = 4x(x-1)(x+1)$

$f'(x) = 0 \Rightarrow x = 0,\ x = 1,\ x = -1$

CRIT #: $x = 1$

END PTS: $x = 0$

$x = 2$

x	$f(x)$	
1	0	← ABS MIN
0	1	
2	9	← ABS MAX

12. (8 points) Use calculus techniques to find the absolute minimum and maximum values of $g(x) = \frac{1}{2}x - x^{2/3}$ on $[-1, 4]$.

$$g'(x) = \frac{1}{2} - \frac{2}{3}x^{-1/3}$$

$$= \frac{1}{2} - \frac{2}{3\sqrt[3]{x}}$$

$g'(x) = 0 \Rightarrow \dfrac{1}{2} = \dfrac{2}{3\sqrt[3]{x}}$

$\sqrt[3]{x} = \dfrac{4}{3}$

$x = \left(\dfrac{4}{3}\right)^3 = \dfrac{64}{27} \approx 2.37$

$g'(x)$ DNE $\Rightarrow x = 0$

CRIT #'S: $x = 0,\ x = \dfrac{64}{27}$

ENDPTS: $x = -1,\ x = 4$

x	$g(x)$	
0	0	← ABS MAX
$64/27$	$-16/27$	
-1	$-\frac{3}{2}$	← ABS MIN
4	-0.5198	

13. (10 points) Use calculus techniques to find open intervals on which

$$f(x) = 2x^3 - 9x^2 + 12x - 5$$

is increasing/decreasing. Also identify all relative extreme values.

$f'(x) = 6x^2 - 18x + 12$

$= 6(x^2 - 3x + 2) = 6(x-1)(x-2)$.

THE ONLY CRITICAL #'S SATISFY

$f'(x) = 0$

$x = 1,\ x = 2$

SIGNS OF $f'(x)$: $+$ | $-$ | $+$ (at 1 and 2)

f IS: INC (1) DEC (2) INC

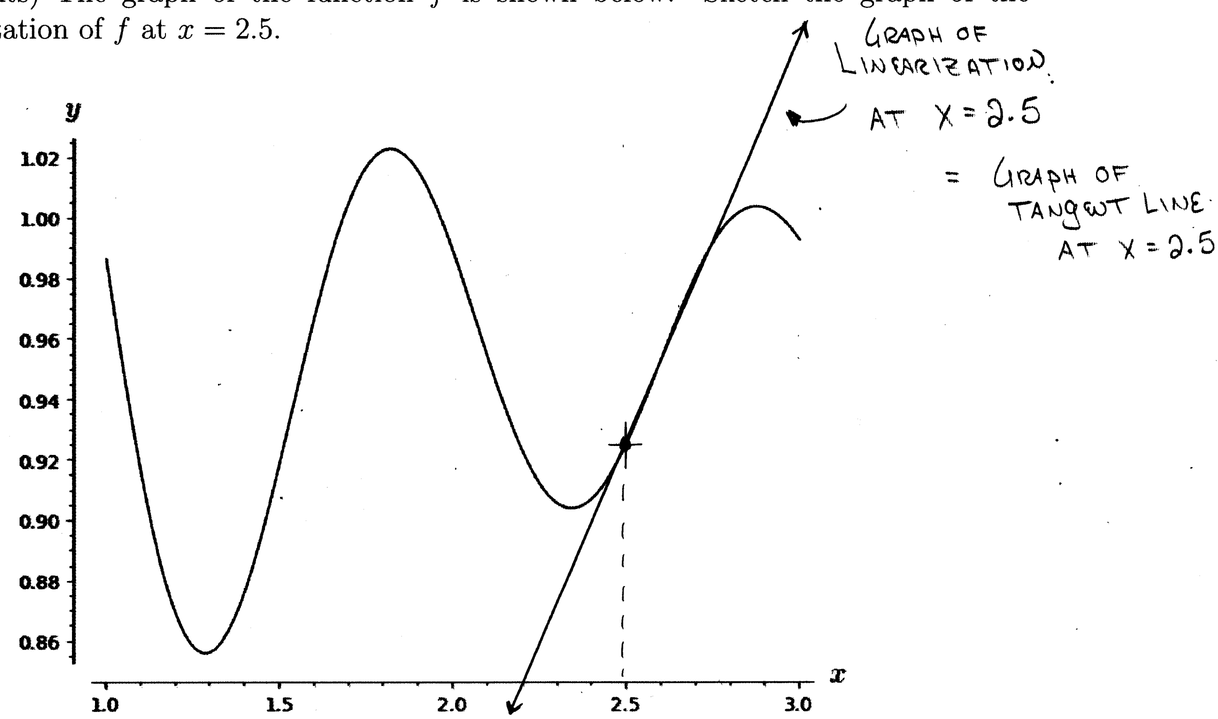
f IS INCREASING ON $(-\infty, 1) \cup (2, \infty)$.

f IS DECREASING ON $(1, 2)$.

$f(1) = 0$ IS A REL. MAX

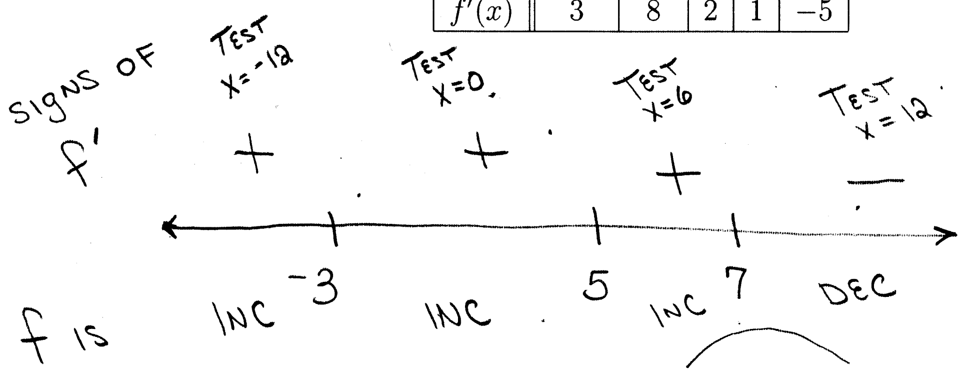
$f(2) = -1$ IS A REL MIN.

14. (2 points) The graph of the function f is shown below. Sketch the graph of the linearization of f at $x = 2.5$.



15. (6 points) The functions $f(x)$ and $f'(x)$ are defined for all x. Furthermore, $f'(x)$ has exactly three zeros: $x = -3$, $x = 5$, and $x = 7$. Use this and the information below to find the locations (x-values) of all relative extrema.

x	-12	-6	0	6	12
$f'(x)$	3	8	2	1	-5



RELATIVE MAX AT X=7.

OTHER CRIT #'S DO NOT GIVE EXTREMA.