# Math 131 - Final Exam
May 12, 2022

Name key

Score ____________

**Show all work to receive full credit**. For each problem, place your final answer in the box provided. Each problem is worth 5 points–up to 2 points for the answer and up to 3 points for the supporting work or explanation. Derivatives need not be simplified.

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1. Determine the limit. Show analytically (not with a graph or table) how you got your answer.

$$\lim_{x\to 0} \frac{1+\tan x}{x^2(x-7)}$$

THE LIMIT HAS THE FORM OF $\frac{1}{0}$, SO THERE IS SOME KIND OF INFINITE LIMIT.

FOR $X \approx 0^+$, THE FRACTION IS $\frac{+}{-} = -$

FOR $X \approx 0^-$, THE FRACTION IS $\frac{+}{-} = -$

THE LIMIT IS $-\infty$.

$$\boxed{-\infty}$$

2. The function $f(x) = \dfrac{x^2+x}{x^2-2x}$ has two discontinuities. Find and classify them.

$$f(x) = \underbrace{\frac{x(x+1)}{x(x-2)}}_{} = \frac{x+1}{x-2}, \quad x \neq 0$$

DISCONTS AT $X=0$, $X=2$

$X=0$ IS REMOVABLE, $X=2$ IS INFINITE. ($\frac{3}{0}$ FORM)

$X=0$ IS A REMOVABLE DISCONT.

$X=2$ IS AN INFINITE DISCONT.

3. Use a table of numerical values to estimate the limit: $\lim_{x \to 1^+} \left( \frac{3}{\ln x} - \frac{2}{x-1} \right)$

| $X$ | $\frac{3}{\ln x} - \frac{2}{x-1}$ |
|---|---|
| 1.1 | 11.47618 |
| 1.01 | 101.49751 |
| 1.001 | 1001.49975 |
| 1.0001 | 10001.49998 |

$\boxed{\text{LIMIT IS } +\infty.}$

4. Determine the limit. Use algebraic techniques (not a graph, table, or L'Hôpital's rule) to show how you got your answer.

0/6

$$\lim_{x \to 16} \left( \frac{16 - x}{4 - \sqrt{x}} \right) \cdot \frac{4+\sqrt{x}}{4+\sqrt{x}}$$

$$= \lim_{x \to 16} \frac{\cancel{(16-x)}(4+\sqrt{x})}{\cancel{(16-x)}} = 8$$

$\boxed{8}$

5. Find $\frac{dy}{dx}$ if $y = \frac{\sqrt{x} + 5x^3}{\cos x}$.

QUOTIENT RULE

$$\boxed{\frac{dy}{dx} = \frac{(\cos x)(\frac{1}{2}x^{-1/2} + 15x^2) - (\sqrt{x} + 5x^3)(-\sin x)}{\cos^2 x}}$$

6. Let $f(x) = x^2 - x$. Write $f'(x)$ in the box, then use the limit definition of derivative to obtain your answer.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left[(x+h)^2 - (x+h)\right] - \left[x^2 - x\right]}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x - h - x^2 + x}{h} = \lim_{h\to 0} \frac{2xh + h^2 - h}{h}$$

$$= \lim_{h\to 0} (2x + h - 1)$$

$$= \boxed{2x - 1}$$

$$f'(x) = 2x - 1$$

7. Compute $g'(1/2)$ if $g(x) = 1 + 5\cot(\pi x^2)$.

$$g'(x) = 5\left(-\csc^2(\pi x^2)\right)(2\pi x)$$

$$= -10\pi x \csc^2(\pi x^2)$$

$$g'\left(\tfrac{1}{2}\right) = -10\pi\left(\tfrac{1}{2}\right)\csc^2\left(\tfrac{\pi}{4}\right) = -5\pi(\sqrt{2})^2$$

$$= -10\pi$$

$$g'\left(\tfrac{1}{2}\right) = -10\pi$$

8. A potato is launched vertically upward with an initial velocity of 100 ft/s from a potato gun at the top of an 85-foot-tall building. The height in feet of the potato after $t$ seconds (measured from the ground) is given by $s(t) = -16t^2 + 100t + 85$. Use calculus to find the maximum height of the potato.

$$S'(t) = -32t + 100$$

$$s'(t) = 0 \Rightarrow t = \frac{100}{32} = 3.125$$

$$S(3.125) = -16(3.125)^2 + 312.5 + 85 = 241.25 \text{ FT}$$

$$\boxed{241.25 \text{ FT}}$$

9. Let $h(x) = \cos^{-1}(3x) - \tan^{-1}(x^3)$. Determine $h'(x)$.

$$h'(x) = \frac{-3}{\sqrt{1-(3x)^2}} - \frac{3x^2}{1+(x^3)^2}$$

$$\boxed{h'(x) = \frac{-3}{\sqrt{1-9x^2}} - \frac{3x^2}{1+x^6}}$$

10. Compute $f'(0)$ if $f(x) = e^{\sin(2x)}$.

$$f'(x) = e^{\sin(2x)} \cdot \cos(2x) \cdot 2$$

$$f'(0) = e^0 \cdot \cos(0) \cdot 2 = 2$$

$$\boxed{2}$$

11. The gamma function, $\Gamma(x)$, is a special function that arises in many applications. Use the values,

$$\Gamma(1) = 1 \qquad \text{and} \qquad \Gamma'(1) = -0.577215665,$$

to find the linearization for $\Gamma$ at $x = 1$. Then use your linearization to approximate $\Gamma(1.05)$.

$$L(x) = \Gamma(1) + \Gamma'(1)(x-1) = 1 - 0.577215665\,(x-1)$$

$$\boxed{\begin{array}{c} L(x) = 1 - 0.577215665(x-1) \\ \Gamma(1.05) \approx L(1.05) \approx 0.971 \end{array}}$$

12. Evaluate the limit: $\displaystyle\lim_{x \to 0} \frac{\tan^{-1} x}{\pi x}$ $\quad 0/0$

$$= \lim_{x \to 0} \frac{\frac{1}{1+x^2}}{\pi} = \frac{1}{\pi}$$

$$\boxed{1/\pi}$$

13. Find the critical number(s) of $f(x) = \dfrac{1}{2x^3 - 3x^2}$.

$$f'(x) = \frac{-(6x^2 - 6x)}{(2x^3 - 3x^2)^2} = \frac{6x(1-x)}{x^4(2x-3)^2} = \frac{6(1-x)}{x^3(2x-3)^2}$$

$f'(x) = 0 \Rightarrow x = 1$, $f'(x)$ DNE $\Rightarrow \underbrace{x = 0,\ x = \tfrac{3}{2}}_{}$

NEITHER IS IN THE DOMAIN OF $f$.

$$\boxed{x = 1 \text{ IS THE ONLY CRITICAL NUMBER.}}$$

14. The function $f$ has critical numbers $x = 0$, $x = 1/2$, and $x = 1$. Furthermore, its second derivative is given by $f''(x) = 12x^2 - 12x + 2$. Use the second derivative to determine whether the critical numbers give relative maxima or minima.

$f''(0) = 2 > 0 \Rightarrow$ CU $\Rightarrow$ MIN

$f''(\frac{1}{2}) = -1 < 0 \Rightarrow$ CD $\Rightarrow$ MAX

$f''(1) = 2 > 0 \Rightarrow$ CU $\Rightarrow$ MIN

$\boxed{\begin{array}{l} f(0) \text{ AND } f(1) \text{ ARE REL. MIN.} \\ f(\frac{1}{2}) \text{ IS A REL MAX.} \end{array}}$

15. Use calculus techniques to find the absolute extreme values of $f(x) = (x+2)^2e^{-x}$ on $[-3, 1]$. (Remember that $e^{-x} > 0$ for all $x$.)

$$f'(x) = 2(x+2)e^{-x} - (x+2)^2e^{-x} = e^{-x}(2x+4 - x^2 - 4x - 4)$$
$$= e^{-x}(-x^2 - 2x) = -(x^2+2x)e^{-x}$$

$f'(x) = 0 \Rightarrow x = 0,\ x = -2$

| $x$ | $f(x)$ | |
|---|---|---|
| 0 | 4 | |
| -2 | 0 | $\leftarrow$ ABS MIN |
| -3 | $e^3$ | $\leftarrow$ ABS MAX |
| 1 | $9/e$ | |

$\boxed{\begin{array}{l} \text{ABS MIN IS } f(-2) = 0 \\ \text{ABS MAX IS } f(-3) = e^3 \end{array}}$

16. Evaluate the indefinite integral: $\int \left( \frac{2}{x} + e^x - \frac{1}{x^2} \right) dx$

$$2 \int \frac{1}{x}\, dx = 2 \ln|x| + C$$

$$\int e^x\, dx = e^x + C$$

$$-\int x^{-2}\, dx = -\left(-1x^{-1}\right) + C$$

$$\boxed{2 \ln|x| + e^x + \frac{1}{x} + C}$$

17. Let $f(x) = \ln x$. Use 5 subintervals of equal length and subinterval midpoints (for the $c_k$'s) to compute a Riemann sum for $f$ on $[1, 2]$.

$$\Delta x = \frac{2-1}{5} = \frac{1}{5} = 0.2$$

PARTITION IS $1 < 1.2 < 1.4 < 1.6 < 1.8 < 2$

$c_1 = 1.1$ $c_2 = 1.3$ $c_3 = 1.5$ $c_4 = 1.7$ $c_5 = 1.9$

$$\text{RIEMANN SUM} = 0.2\left[\ln 1.1 + \ln 1.3 + \ln 1.5 + \ln 1.7 + \ln 1.9\right]$$

$$\approx (0.2)(1.93562169)$$

$$\approx 0.38712$$

$$\boxed{\approx 0.38712}$$

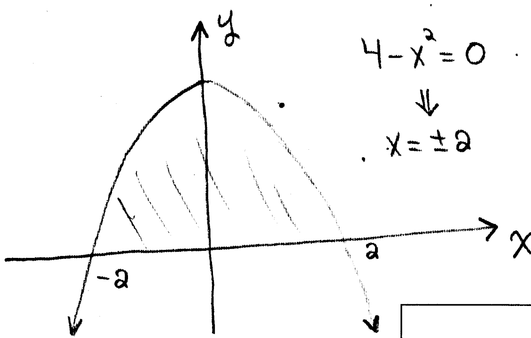
18. Evaluate the definite integral: $\int_1^9 \sqrt{x}\,dx$ $= \int_1^9 x^{1/2}\,dx = \frac{2}{3}x^{3/2}\Big|_1^9$

$= \frac{2}{3}\left[9^{3/2} - 1^{3/2}\right]$

$= \frac{2}{3}\left[27-1\right] = \frac{52}{3}$

$\boxed{\frac{52}{3} = 17.\overline{3}}$

19. Use a definite integral to find the area of the bounded region between the graph of $y = 4 - x^2$ and the $x$-axis.




$4 - x^2 = 0$

$\Downarrow$

$x = \pm 2$

$\int_{-2}^{2} (4-x^2)\,dx = 4x - \frac{1}{3}x^3\Big|_{-2}^{2}$

$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$

$= 16 - \frac{16}{3}$

$\boxed{\frac{32}{3}}$

20. Use a $u$-substitution to evaluate the definite integral: $\int_0^1 3x(x^2+1)^4\,dx$

$u = x^2 + 1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = 2$

$\frac{3}{2}\int_1^2 u^4\,du = \frac{3}{10}u^5\Big|_1^2$

$= \frac{3}{10}\left[32 - 1\right] = \frac{93}{10}$

$\boxed{\frac{93}{10} = 9.3}$