Math 131 - Test 2
March 8, 2023

Name key Score ______

Show all work to receive full credit. Supply explanations where necessary. Unless otherwise indicated, use differentiation rules for all derivatives and do not simplify. You may get partial credit on multiple choice problems if you supply correct work or explanations.

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1. (8 points) Let $f(x) = 3x - 7x^2$. Use the **limit definition of the derivative** to determine $f'(x)$. Show all work.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left[3(x+h) - 7(x+h)^2\right] - \left[3x - 7x^2\right]}{h}$$

$$= \lim_{h\to 0} \frac{3x + 3h - 7x^2 - 14xh - 7h^2 - 3x + 7x^2}{h} = \lim_{h\to 0} \frac{3h - 14xh - 7h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(3 - 14x - 7h)}{h} = \lim_{h\to 0} (3 - 14x - 7h)$$

$$= \boxed{3 - 14x}$$

2. (4 points) Once again, let $f(x) = 3x - 7x^2$. Use differentiation rules to determine $f'(x)$. Then find an equation of the line tangent to the graph of f at the point where $x = 1$.

$f'(x) = 3 - 14x$

Slope $= m = 3 - 14(1) = -11$

Point: $x = 1 \Rightarrow y = f(1) = -4$

$(1, -4)$

Tan. Line

$y + 4 = -11(x-1)$

or

$y = -11x + 7$

3. (5 points) Let $g(x) = 5 + 6\sqrt[3]{x^2}$.

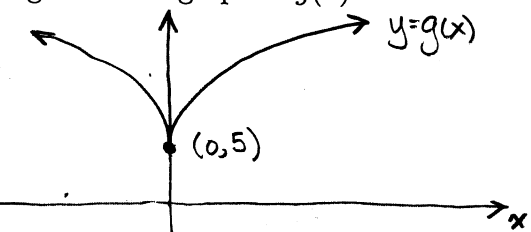
(a) Determine $g'(x)$.

$$g(x) = 5 + 6x^{2/3} \Rightarrow g'(x) = 6\left(\tfrac{2}{3}\right)x^{-1/3} = \frac{4}{\sqrt[3]{x}}.$$

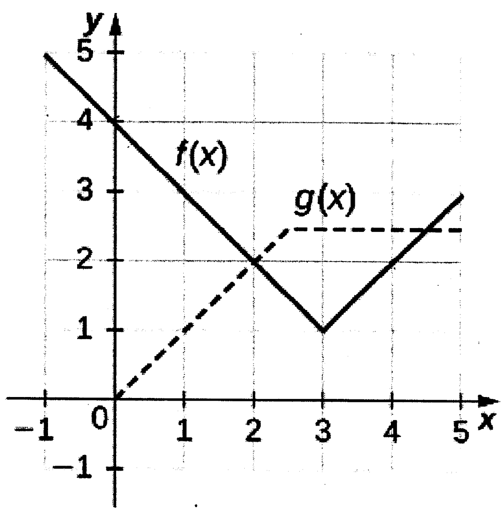
(b) Which one of the following best describes the line tangent to the graph of $g(x)$ at the point $(0, 5)$?

i. The tangent line has slope 5.
ii. The tangent line does not exist.
iii. The tangent line is horizontal.
iv. The tangent line is vertical.

SHARP PT



4. (10 points) The graphs of functions f and g are shown below. Use the graphs to estimate the function values required for each part of this problem.



(a) Determine $g'(4)$.

$g'(4) = 0$ (HORIZONTAL)

(b) Explain why $f'(3)$ does not exist.

THE GRAPH HAS A SHARP PT AT $X=3$. SLOPE FROM RIGHT $\neq$ SLOPE FROM LEFT

(c) Let $h(x) = xf(x)$. Determine $h'(2)$.

$h'(x) = f(x) + x f'(x)$ (PRODUCT RULE) $h'(2) = f(2) + 2f'(2)$

$= 2 + 2(-1) = \boxed{0}$

(d) Let $h(x) = \dfrac{g(x)}{f(x)}$. Determine $h'(1)$.

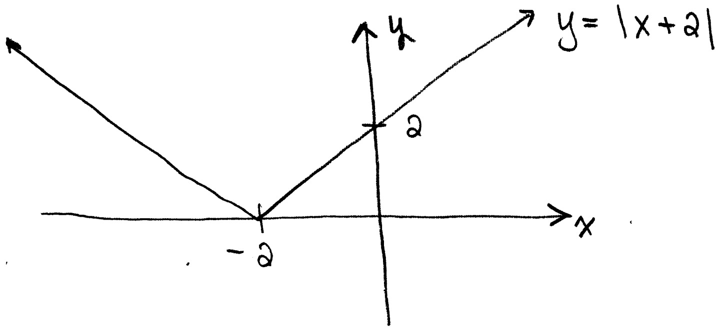
QUOTIENT RULE...

$$h'(x) = \frac{f(x)g'(x) - g(x)f'(x)}{[f(x)]^2} \Rightarrow h'(1) = \frac{f(1)g'(1) - g(1)f'(1)}{[f(1)]^2}$$

$$= \frac{(3)(1) - (1)(-1)}{3^2} = \boxed{\frac{4}{9}}$$

5. (4 points) Let $F(x) = |x+2|$.

(a) Sketch the graph of $F(x)$.



(b) Which one of the following best describes the line tangent to the graph of $F(x)$ at the point $(-2, 0)$?

i. The tangent line is horizontal.
ii. The tangent line is vertical.
iii. The tangent line does not exist.
iv. The tangent line has slope 1.

SHARP PT AT $X = -2$

6. (20 points) Differentiate. Do not simplify.

(a) $\frac{d}{dx}\left(x^{5/7} + 2x^{-1} - \frac{1}{x^2}\right) = \frac{d}{dx}\left(x^{5/7} + 2x^{-1} - x^{-2}\right)$

$$= \frac{5}{7}x^{-2/7} - 2x^{-2} + 2x^{-3}$$

(b) $\frac{d}{dt}\left(\frac{t\sin t}{\cot t}\right) = \frac{(\cot t)(t\cos t + \sin t) - (t\sin t)(-\csc^2 t)}{\cot^2 t}$

(c) $\frac{d}{dx}\tan(\pi x^2) = \sec^2(\pi x^2)(2\pi x)$

(d) $\frac{d}{dt}\left[t^2(9t-4)^8\right] = 2t(9t-4)^8 + t^2(8)(9t-4)^7(9)$

$$= 2t(9t-4)^8 + 72t^2(9t-4)^7$$

7. An object is launched vertically so that its height (in feet) after t seconds is given by

$$s(t) = -16t^2 + 48t + 160.$$

Include units with your answer for each part of this problem.

(a) (3 points) Determine the average rate of change the object's height over the interval from $t = 1$ to $t = 3$.

$$\frac{\Delta s}{\Delta t} = \frac{s(3)-s(1)}{3-1} = \frac{160-192}{2} = \frac{-32}{2} = -16 \text{ FT/SEC}$$

(b) (3 points) Determine the object's velocity at time $t = 3$.

$$s'(t) = -32t + 48 \qquad s'(3) = -96 + 48 = -48 \text{ FT/SEC}$$

(c) (2 points) What is the acceleration of the object?

$$s''(t) = -32 \text{ FT/SEC}^2$$

(d) (3 points) When does the object hit the ground?

$$s(t) = -16(t^2 - 3t - 10)$$
$$= -16(t-5)(t+2) = 0 \quad \Rightarrow \quad t = 5 \text{ SEC}$$

(e) (2 points) What is the object's speed when it hits the ground?

$$s'(5) = -32(5) + 48$$
$$= -112$$

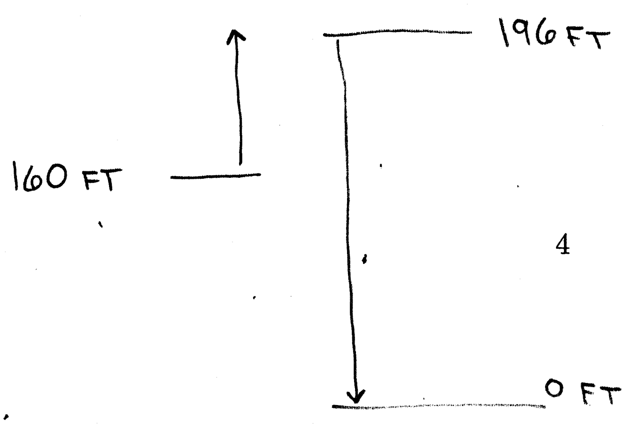
$$\text{SPEED} = |-112|$$
$$= 112 \text{ FT/SEC}$$

(f) (4 points) Determine the object's maximum height.

$$s'(t) = 0 \Rightarrow -32t + 48 = 0$$
$$\Rightarrow t = \frac{48}{32} = 1.5 \text{ SEC}$$

$$s(1.5) = 196 \text{ FT}$$

(g) (2 points) What is the overall length of the object's path?



$$196 + 36 = 232 \text{ FT}$$

8. (12 points) The graph of the equation $x^3 + xy^2 = 3x^2 - y^2$ is called a *trisectrix of Maclaurin.*

(a) Use implicit differentiation to find a formula for dy/dx.

$$\frac{d}{dx}\left(x^3 + xy^2\right) = \frac{d}{dx}\left(3x^2 - y^2\right)$$

$$3x^2 + y^2 + 2xy\frac{dy}{dx} = 6x - 2y\frac{dy}{dx}$$

$$2xy\frac{dy}{dx} + 2y\frac{dy}{dx} = 6x - 3x^2 - y^2$$

$$\boxed{\frac{dy}{dx} = \frac{6x - 3x^2 - y^2}{2xy + 2y}}$$

(b) Use dy/dx to compute the slope of the graph at the point $(1, -1)$. Then determine an equation for the tangent line at $(1, -1)$.

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(1,-1)} = \frac{6-3-1}{-2-2} = \frac{2}{-4}$$

TAN LINE $\boxed{y + 1 = -\frac{1}{2}(x-1) \quad \text{OR} \quad y = -\frac{1}{2}x - \frac{1}{2}}$

9. (4 points) Find the slope of the line tangent to the graph of $y = \tan^{-1}(3x + 1)$ at the point where $x = 2$.

$$\frac{dy}{dx} = \frac{1}{(3x+1)^2 + 1} \cdot 3 = \frac{3}{(3x+1)^2 + 1}$$

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{3}{49+1} = \boxed{\frac{3}{50}}$$

10. (6 points) For $x \geq 1$, let $g(x) = x^2 - 2x + 5$. The function g has an inverse.

(a) Determine the value of $g^{-1}(8)$.

$$g^{-1}(8) = x \iff g(x) = 8$$

$$x^2 - 2x + 5 = 8$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$x = 3$

$$\boxed{g^{-1}(8) = 3}$$

(b) Now find $(g^{-1})'(8)$.

$$(g^{-1})'(8) = \frac{1}{g'(g^{-1}(8))} = \frac{1}{g'(3)} = \boxed{\frac{1}{4}}$$

$$g'(x) = 2x - 2 \qquad g'(3) = 4$$

11. (4 points) Find the instantaneous rate of change of $f(x) = e^{x^2+1}$ at the point where $x = 1$. Write your answer in decimal form, rounded to the nearest thousandth.

$$f'(x) = e^{x^2+1} \cdot 2x$$

$$f'(1) = 2e^2 \approx \boxed{14.778}$$

12. (4 points) Find $f'(\pi/4)$ when $f(x) = \ln(\cos x)$.

$$f'(x) = \frac{1}{\cos x} \cdot (-\sin x) = \frac{-\sin x}{\cos x}$$

$$f'\left(\frac{\pi}{4}\right) = \frac{-\sin(\pi/4)}{\cos(\pi/4)} = \frac{-\sqrt{2}/2}{\sqrt{2}/2} = \boxed{-1}$$