Math 131 - Test 3

April 12, 2023

Name Key

Score ______

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) Use logarithmic differentiation to find dy/dx when $x = 3$.

$$y = \frac{x^2(x-2)^5}{\sqrt{x^2+16}}$$

$$\ln y = \ln\left(\frac{x^2(x-2)^5}{\sqrt{x^2+16}}\right)$$

$$= 2\ln x + 5\ln(x-2) - \tfrac{1}{2}\ln(x^2+16)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{5}{x-2} - \frac{1}{2}\cdot\frac{2x}{x^2+16}$$

$$\frac{dy}{dx} = \left(\frac{x^2(x-2)^5}{\sqrt{x^2+16}}\right)\left(\frac{2}{x} + \frac{5}{x-2} - \frac{x}{x^2+16}\right)$$

When $x=3$,

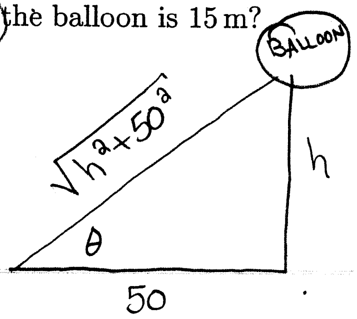
$$\frac{dy}{dx} = \left(\frac{9}{5}\right)\left(\frac{2}{3} + 5 - \frac{3}{25}\right) = \boxed{\frac{1248}{125} = 9.984}$$

2. (4 points) Evaluate the limit: $\displaystyle\lim_{x\to-\infty}\left(\frac{x^4-4x^3+1}{2-2x^2-7x^4}\right)\cdot\frac{\frac{1}{x^4}}{\frac{1}{x^4}}$

$$= \lim_{x\to-\infty}\frac{1-\frac{4}{x}+\frac{1}{x^4}}{\frac{2}{x^4}-\frac{2}{x^2}-7} = \frac{1-0+0}{0-0-7} = \boxed{-\frac{1}{7}}$$

3. (8 points) A balloon is rising straight up from a level field at a rate of 2 m/sec. An observer 50 m downrange is watching the balloon's ascent. How fast is the angle of elevation from the observer to the balloon changing at the moment when the height of the balloon is 15 m?

LECTURE 19 EXAMPLE 5



$$\frac{dh}{dt} = 2$$

FIND $\frac{d\theta}{dt}$ WHEN $h = 15$.

$$\tan\theta = \frac{h}{50}$$

$$\frac{d}{dt}\left(\tan\theta\right) = \frac{d}{dt}\left(\frac{h}{50}\right)$$

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{50}\cdot\frac{dh}{dt}$$

$h = 15$

$\Rightarrow \sqrt{h^2+50^2}$

$= \sqrt{2725}$

WHEN $h = 15$...

$$\sec^2\theta = \frac{2725}{2500}$$

$$\frac{d\theta}{dt} = \frac{2}{50}\cdot\frac{2500}{2725}$$

$$= 0.0367$$

RADIANS/SEC

4. (6 points) Some values of $f(x)$ and $f'(x)$ are given in the table below.

x	-1	0	1	2
$f(x)$	-0.5737	0.7027	1.1044	11.2234
$f'(x)$	0.0740	0.2082	0.7423	171.4132

(a) Determine the linearization of f at $x = 1$.

$$L(x) = f(1) + f'(1)(x-1)$$

$$\boxed{L(x) = 1.1044 + 0.7423(x-1)}$$

(b) Use the linearization you found above to approximate $f(0.925)$.

$$f(0.925) \approx L(0.925)$$

$$= 1.1044 + 0.7423(-0.075)$$

$$\approx \boxed{1.0487}$$

5. (6 points) Let $y = \dfrac{1}{2x+1}$. Use differentials to approximate Δy as x changes from $x = 1$ to $x = 1.25$.

$$dy = \frac{-2}{(2x+1)^2}\,dx$$ (QUOTIENT RULE)

$$\Delta y \approx \frac{-2}{(2x+1)^2}\,\Delta x$$

$x = 1$, $\Delta x = 0.25$

$$\Delta y \approx \frac{-2}{9}(0.25)$$

$$\frac{-0.5}{9} = -\frac{1}{18} = -0.0\overline{5}$$

6. (8 points) Use calculus techniques to determine the absolute minimum and maximum values of $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$.

$f'(x) = 2x - 2x^{-1/3}$

$f'(x)$ DNE WHEN $X = 0$ BUT THAT'S AN ENDPT.

$f'(x) = 0 \Rightarrow X = X^{-1/3}$

$X = \dfrac{1}{\sqrt[3]{X}}$

$X^{4/3} = 1$

$X = 1$

X	f(x)	
0	0	← ABS MAX
2	-0.7622...	
1	-2	← ABS MIN

7. (8 points) Evaluate the limit: $\lim_{x \to 0} \left(\dfrac{e^x - 4x^2 - 2 + e^{-x}}{x^2} \right)$ 0/0 MORE WORK.

$$= \lim_{X \to 0} \frac{e^X - 8x - e^{-X}}{2x} =$$ 0/0 MORE WORK.

$$\lim_{X \to 0} \frac{e^X - 8 + e^{-X}}{2} = \frac{-6}{2} = \boxed{-3}$$

8. (8 points) The function $f(x) = 5x^{1/3} - x^{5/3}$ has exactly three critical numbers: $x = -1$, $x = 0$, and $x = 1$. Use calculus techniques to identify all relative extreme values of f.

$f'(x) = \frac{5}{3}x^{-2/3} - \frac{5}{3}x^{2/3}$

SIGNS OF f': TEST $x=-2$: $-$; TEST $x=-0.5$: $+$; TEST $x=0.5$: $+$; TEST $x=2$: $-$

-1 $\quad$ 0 $\quad$ 1

f IS: DEC $\quad$ INC $\quad$ INC $\quad$ DEC

$$\boxed{\begin{array}{l} f(-1) = -4 \text{ IS A REL MIN.} \\ f(1) = 4 \text{ IS A REL MAX.} \end{array}}$$

9. (8 points) Let $f(x) = x^4 - 2x^3 - 12x^2 + 36x + 2$. Find open intervals on which the graph of f is concave up/down. Also identify all inflection points.

$f'(x) = 4x^3 - 6x^2 - 24x + 36$

$f''(x) = 12x^2 - 12x - 24$

$f''(x) = 12(x-2)(x+1) = 0$

$x = 2,\ x = -1$

SIGNS OF f'': $+$ $\quad$ $x=-1$ $\quad$ $-$ $\quad$ $x=2$ $\quad$ $+$

GRAPH IS: CU $\quad$ CD $\quad$ CU

GRAPH IS CONCAVE UP ON $(-\infty,-1) \cup (2,\infty)$ AND CONCAVE DOWN ON $(-1,2)$.

$(-1,-43)$ AND $(2,26)$ ARE INFLECTION POINTS.

10. (6 points) The graph of $y = \dfrac{3x-2}{\sqrt{4x^2+5}}$ has two horizontal asymptotes. Find either one of them. Show all work.

$$\lim_{x\to\infty} \frac{3x-2}{\sqrt{4x^2+5}} \cdot \frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}} = \lim_{x\to\infty} \frac{3-\frac{2}{x}}{\sqrt{4+\frac{5}{x^2}}} = \frac{3}{\sqrt{4}} = \frac{3}{2}$$

$$\boxed{y = \frac{3}{2}}$$

THE OTHER H.A. IS $y = -\frac{3}{2}$.

11. (10 points) Tell whether each statement is true or false.

(a) F L'Hôpital's rule can be used to evaluate a limit involving any kind of indeterminate form.

$\frac{0}{0}$ OR $\frac{\infty}{\infty}$

(b) T If $f'(5) = 0$ and $f''(5) = 10$, then $f(5)$ is a relative minimum.

(c) F Suppose that f is a function for which $f''(x) = x^4$. The graph of f has an inflection point at $x = 0$.

SIGNS OF f'' $+ \mid +$ $x=0$

(d) F Every absolute extreme value is also a relative extreme value.

(e) F If $y = \sin x$, then $dy = \cos x$.

$dy = \cos x \, dy$

12. (6 points) Find the critical numbers of $f(x) = \dfrac{4x^2 - 11x + 9}{x}$. Also, explain why $x = 0$ is not a critical number.

$$f'(x) = \frac{x(8x-11) - (4x^2-11x+9)(1)}{x^2}$$

$$= \frac{4x^2-9}{x^2} = 0$$

$$\Rightarrow x^2 = \frac{9}{4} \Rightarrow \boxed{x = \pm\frac{3}{2}}$$

$f'(x)$ DNE AT $x=0$, BUT $x=0$ IS NOT IN THE DOMAIN OF f

13. (4 points) Suppose k is any positive integer. What is the value of the limit $\lim\limits_{x\to\infty} \dfrac{x^k}{e^x}$? How do you know? (If you need help, think about the cases when $k = 1$ or $k = 2$.)

$\lim\limits_{x\to\infty} \dfrac{x^k}{e^x}$ HAS THE INDETERMINATE FORM $\dfrac{\infty}{\infty}$.

USING L'HÔPITAL'S RULE k TIMES TO GET

$$\lim_{x\to\infty} \frac{x^k}{e^x} = \lim_{x\to\infty} \frac{k\cdot(k-1)\cdot(k-2)\cdots(1)}{e^x}$$

FORM $\dfrac{\#}{\infty}$

$$= 0$$

Math 131 - Test 3 (TH)

April 12, 2023

Name key

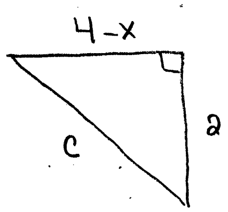
Score

Show all work to receive full credit. Supply explanations where necessary. This problem is due April 17.

1. (10 points) In this problem, you will use calculus techniques to optimize a function in an application. **The Problem:** Starting a point A, a company must lay cable to point B. It is 2 times more expensive to lay the cable through the field than along the road. Referring the the figure, you will find the x-value that minimizes the overall cost.

   (a) The length of the cable through the field is $\sqrt{4+(4-x)^2}$. Explain where this expression comes from. Also expand the polynomial under the radical and combine like terms.

   Pythag Theorem

   $$c^2 = (4-x)^2 + 2^2 = 20 - 8x + x^2$$

   $$c = \sqrt{20 - 8x + x^2}$$

   (b) The cost of laying the cable along the road is k dollars per mile. Therefore, the cost through the field is $2k$ dollars per mile. This makes the overall cost of the project

   $$C(x) = kx + 2k\sqrt{20 - 8x + x^2}, \text{ where } 0 \le x \le 4.$$

   Determine $C'(x)$.

   $$C'(x) = k + k\left(20 - 8x + x^2\right)^{-1/2}(2x-8) = k + \frac{(2x-8)k}{\sqrt{20-8x+x^2}}$$

   (c) Determine the critical number of C. The algebra will be a little bit messy. Feel free to use technology to solve the necessary equation. (The critical number does not depend on k–you can just ignore it.)

   $$k + \frac{(2x-8)k}{\sqrt{20-8x+x^2}} = 0 \Rightarrow 2x - 8 = -\sqrt{20-8x+x^2}$$

   $$4x^2 - 32x + 64 = 20 - 8x + x^2$$

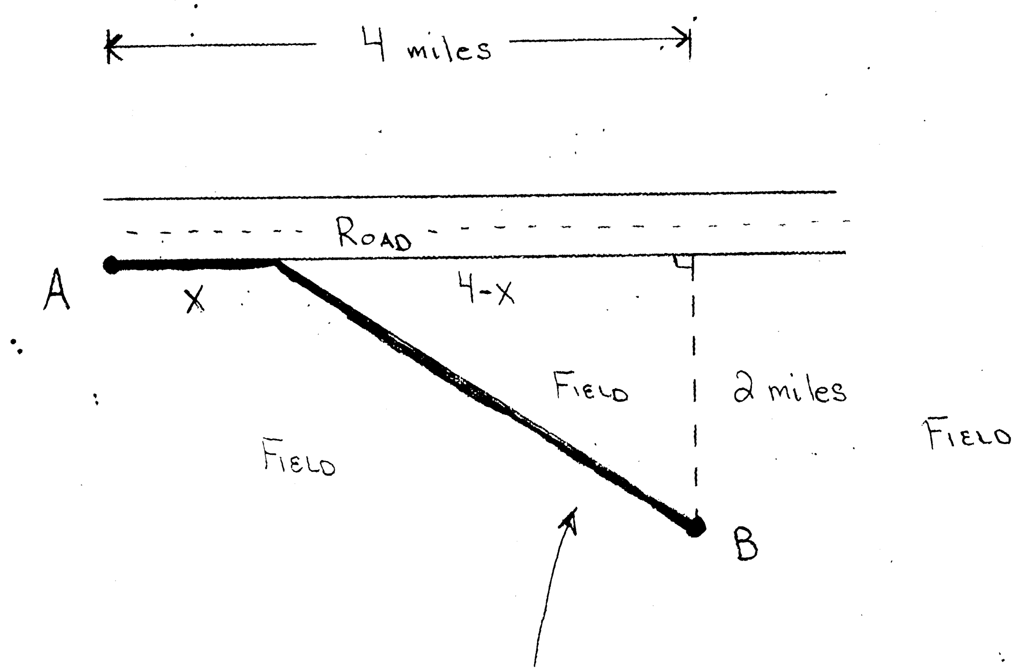
   $$3x^2 - 24x + 44 = 0$$

   $$x = 4 - \frac{2\sqrt{3}}{3} \approx 2.8453$$

   (The other solution is greater than 4.)

   (d) Show that your critical number gives an absolute minimum.

x	$C(x)$	
0	$8.944272k$	← Abs max
4.	$8k$	
2.8453	$7.464102k$	← Abs min.



CABLE IS LAID IN TWO SEGMENTS: ONE ALONG THE ROAD AND ONE THROUGH THE FIELD.