Math 131 - Final Exam

May 8, 2023

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Use algebraic techniques (not a graph, table, or L'Hôpital's rule) to determine each limit. 0/0 MORE WORK

   (a) $\lim_{r \to 1} \frac{\sqrt{r}-1}{r-1} \cdot \frac{\sqrt{r}+1}{\sqrt{r}+1} = \lim_{r \to 1} \frac{\cancel{r-1}}{(\cancel{r-1})(\sqrt{r}+1)} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$

   (b) $\lim_{x \to 6} \frac{(x+3)^2 - 5(x+3) - x^2}{3(x-6)}$ 0/0 MORE WORK

   $= \lim_{x \to 6} \frac{x^2+6x+9-5x-15-x^2}{3(x-6)} = \lim_{x \to 6} \frac{\cancel{x-6}}{3(\cancel{x-6})}$

   $= \frac{1}{3}$

2. (10 points) Use the definition of continuity to explain why f is discontinuous at $x = 5$. Also state the type of discontinuity.

$$f(x) = \begin{cases} 4x+5, & x < 5 \\ x^2 + x\cos(\pi x), & x \geq 5 \end{cases}$$

$\lim_{x \to 5^-} f(x) = 4(5)+5 = 25$

$\lim_{x \to 5^+} f(x) = 5^2 + 5\cos(5\pi)$

$= 25 - 5 = 20$

LIMIT DNE AT $X = 5$.

CANNOT BE CONT. AT $X = 5$.

BECAUSE ONE-SIDED LIMITS EXIST, BUT ARE NOT EQUAL, THE DISCONT. IS A $\boxed{\text{JUMP DISCONTINUITY}}$.

3. (10 points) Let $f(x) = x^2 - 4x$. Use the **limit definition of the derivative** to determine $f'(x)$. Show all work.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left[(x+h)^2-4(x+h)\right]-\left[x^2-4x\right]}{h}$$

$$= \lim_{h\to 0} \frac{x^2+2xh+h^2-4x-4h-x^2+4x}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2-4h}{h} = \lim_{h\to 0}(2x+h-4) = 2x-4$$

$$\boxed{f'(x) = 2x-4}$$

4. (10 points) Use basic differentiation rules to determine each derivative. Do not simplify.

(a) $\dfrac{d}{dx}\left(\dfrac{\tan^{-1} x}{1+2x+x^2}\right)$

$$= \frac{(1+2x+x^2)\left(\frac{1}{1+x^2}\right)-(\tan^{-1}x)(2+2x)}{(1+2x+x^2)^2}$$

(b) $\dfrac{d}{dx}\left[e^{-5x^2}\cot x\right]$

$$= -10x\,e^{-5x^2}\cot x - e^{-5x^2}\csc^2 x$$

5. (10 points) Let $f(x) = \sin(x) + \ln(x^2)$. Find the linearization of f at $x = 1$. Then use your linearization to approximate $f(0.8)$. Write your answers with all numbers in decimal form, rounded to three places. (Make sure your calculator is in radian mode.)

$$f(1) = \sin(1) + \ln(1) = \sin(1) \approx 0.841$$

$$f'(x) = \cos x + \frac{2}{x}, \quad f'(1) = \cos(1) + 2 \approx 2.540$$

$$\boxed{L(x) = 0.841 + 2.540(x-1)}$$

$$L(0.8) = 0.841 + 2.540(-0.2) = 0.333 \Rightarrow \boxed{f(0.8) \approx 0.333}$$

6. (10 points) Use any analytical method (not a table or graph) to determine each limit.

(a) $\lim\limits_{x\to-\infty} \left(\dfrac{3x^4 - 5x^2 + 8}{3x^2 + 4x^4 + 5x^4}\right) \cdot \dfrac{\frac{1}{x^4}}{\frac{1}{x^4}}$

$$= \lim_{x\to-\infty} \frac{3 - \frac{5}{x^2} + \frac{8}{x^4}}{\frac{3}{x^2} + 4 + 5} = \frac{3}{9} = \boxed{\frac{1}{3}}$$

0/0 More work.

(b) $\lim\limits_{x\to 0} \left(\dfrac{x - \sin 5x}{x^2}\right)$

-4/0 Some kind of inf limit

L'Hopital's Rule

$$\lim_{x\to 0} \frac{1 - 5\cos 5x}{2x}$$

-/+

$$\lim_{x\to 0^+} \frac{1-5\cos 5x}{2x} = -\infty$$

-/-

$$\lim_{x\to 0^-} \frac{1-5\cos 5x}{2x} = +\infty$$

Limit DNE.

7. (10 points) Use calculus techniques to find the **absolute extreme values** of $g(x) = x^4 + 4x^3 - 20x^2$ on the interval $[-2, 3]$.

$g'(x) = 4x^3 + 12x^2 - 40x$

$= 4x(x^2+3x-10)$

$= 4x(x+5)(x-2)$

ENDPTS: $x = -2, x = 3$

$g'(x) = 0 \Rightarrow x = 0, x = -5, x = 2$

CRIT #'s

x	g(x)	
0	0	
2	-32	
-2	-96	← ABS MIN
3	9	← ABS MAX

8. (10 points) Evaluate each definite integral. (You may need to use a substitution.)

(a) $\int_1^2 \frac{1+t+t^2}{t^3}\, dt$

$$= \int_1^2 \left(t^{-3} + t^{-2} + \frac{1}{t}\right) dt$$

$$= -\frac{1}{2}t^{-2} - t^{-1} + \ln|t| \Big|_1^2$$

$$= \left(-\frac{1}{8} - \frac{1}{2} + \ln 2\right) - \left(-\frac{1}{2} - 1 + 0\right) = \frac{7}{8} + \ln 2 \approx 1.568$$

(b) $\int_0^{\pi} (1+\cos x)^3 \sin x\, dx$ $= -\int_2^0 u^3\, du$

$u = 1 + \cos x$

$du = -\sin x\, dx$

$-du = \sin x\, dx$

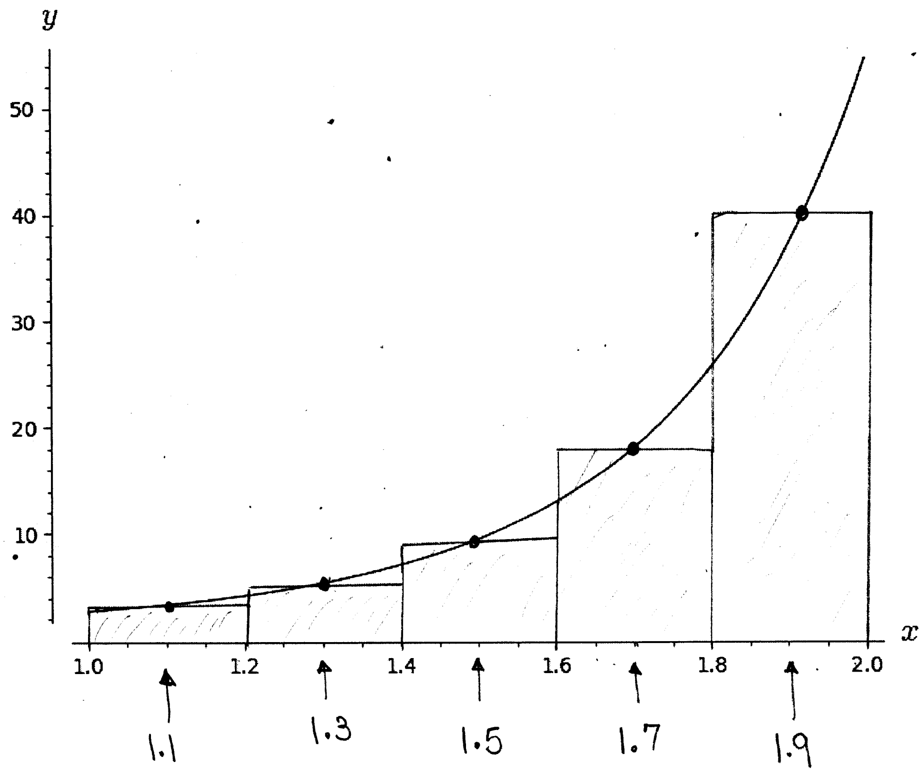
$x = 0 \Rightarrow u = 2$

$x = \pi \Rightarrow u = 0$

$$= \int_0^2 u^3\, du = \frac{1}{4}u^4 \Big|_0^2$$

$$= 4 - 0 = \boxed{4}$$

9. (10 points) The graph of $f(x) = e^{x^2}$ over the interval $[1, 2]$ is shown below. Use five subintervals of equal length and subinterval **midpoints** to compute the corresponding (middle) Riemann sum for f on $[1, 2]$. Once you have computed the Riemann sum, sketch the corresponding rectangles on the graph.



$\Delta x = \frac{2-1}{5} = 0.2$

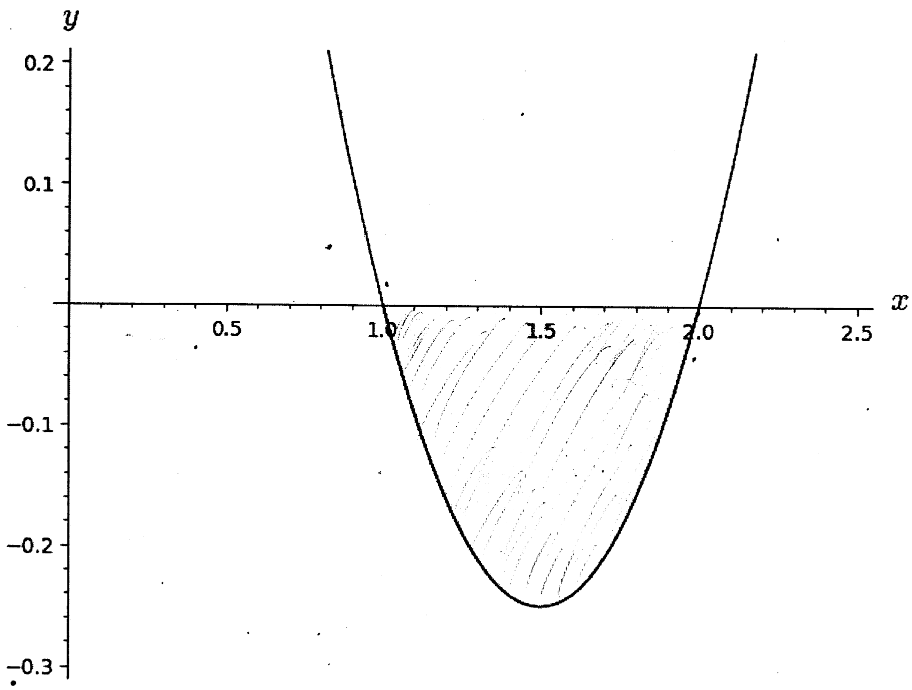
$1 < 1.2 < 1.4 < 1.6 < 1.8 < 2.0$

$c_1 = 1.1, \quad c_2 = 1.3, \quad c_3 = 1.5, \quad c_4 = 1.7, \quad c_5 = 1.9$

Riemann sum $= 0.2\left[e^{(1.1)^2} + e^{(1.3)^2} + e^{(1.5)^2} + e^{(1.7)^2} + e^{(1.9)^2}\right]$

≈ 14.644

10. (10 points) The graph of $y = x^2 - 3x + 2$ is shown below.



(a) By solving $y = 0$ (Show your work!), show algebraically that the x-intercepts of the graph agree with those shown.

$$x^2 - 3x + 2 = 0$$

$$(x-2)(x-1) = 0 \quad \Rightarrow \quad x - 2 = 0 \quad \text{or} \quad x - 1 = 0$$

$$\boxed{x = 2 \quad \text{or} \quad x = 1}$$

(b) Use the fundamental theorem of calculus to find the area of the shaded region.

$$\text{Area} = -\int_1^2 (x^2 - 3x + 2)\,dx = -\left(\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\right)\Bigg|_1^2$$

$$= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x\Bigg|_2^1 = \left(\frac{1}{3} - \frac{3}{2} + 2\right) - \left(\frac{8}{3} - \frac{12}{2} + 4\right)$$

$$= \frac{5}{6} - \frac{2}{3} = \boxed{\frac{1}{6}}$$