Math 131 - Test 2
March 6, 2023

Name key

Score ____________

Show all work to receive full credit. Supply explanations where necessary. Unless otherwise indicated, use differentiation rules for all derivatives and do not simplify. You may get partial credit on multiple choice problems if you supply correct work or explanations.

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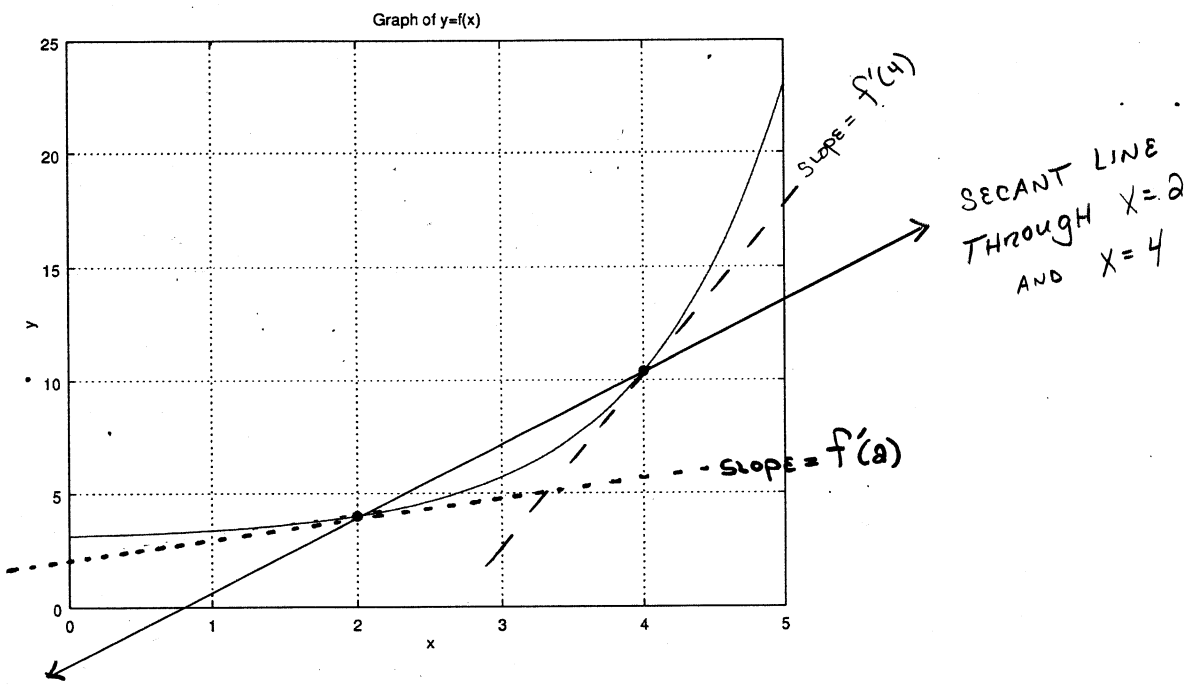
1. (6 points) The graph of $y = f(x)$ is shown below.



(a) Sketch the secant line through the indicated points at $x = 2$ and $x = 4$. Let m be the slope of the secant line through those points. Estimate the value of m.

$X = 2 \Rightarrow y \approx 4$

$X = 4 \Rightarrow y \approx 10.5$

$$m \approx \frac{10.5 - 4}{4 - 2} = \frac{6.5}{2} = 3.25$$

(b) Which number is greatest: m, $f'(2)$, or $f'(4)$?. Explain your reasoning.

$f'(4)$ IS GREATEST. $f'(4)$ = SLOPE OF TAN LINE AT $X=4$. TAN. LINE AT $X=4$ IS STEEPER THAN SECANT LINE AND TAN LINE AT $X=2$.

(c) Which number is least: m, $f'(2)$, or $f'(4)$?. Explain your reasoning.

$f'(2)$ IS LEAST. THE TAN. LINE AT $X=2$ IS THE LEAST STEEP OF THE LINES SHOWN.

2. (8 points) Let $f(x) = 3x - 7x^2$. Use the **limit definition of the derivative** to determine $f'(x)$. Show all work.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left[3(x+h)-7(x+h)^2\right] - \left[3x-7x^2\right]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x}+3h-\cancel{7x^2}-14xh-7h^2-\cancel{3x}+\cancel{7x^2}}{h}$$

$$= \lim_{h\to 0} \frac{3h-14xh-7h^2}{h} = \lim_{h\to 0} \frac{h(3-14x-7h)}{h}$$

$$= \lim_{h\to 0} (3-14x-7h) = \boxed{3-14x}$$

3. (5 points) Once again, let $f(x) = 3x - 7x^2$. Use differentiation rules to determine $f'(x)$. Then find an equation of the line tangent to the graph of f at the point where $x = 1$.

$f'(x) = 3 - 14x$

SLOPE: $m = f'(1) = 3 - 14 = -11$

POINT: $x = 1$, $y = 3 - 7 = -4$

$(1, -4)$

TAN LINE:

$y + 4 = -11(x-1)$

OR

$y = -11x + 7$

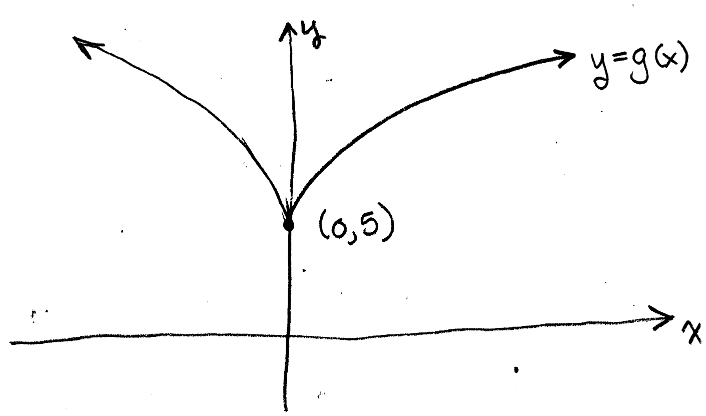
4. (5 points) Let $g(x) = 5 + 6\sqrt[3]{x^2}$.

(a) Determine $g'(x)$.

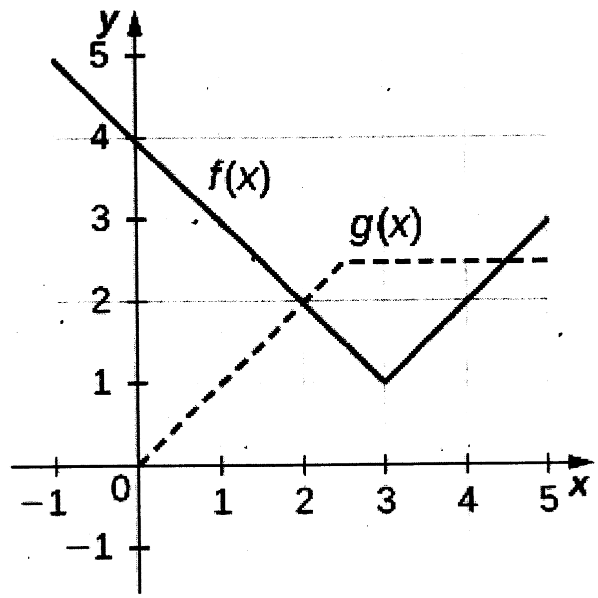
$g(x) = 5 + 6x^{2/3}$ $\quad \boxed{g'(x) = 4x^{-1/3}} = \dfrac{4}{\sqrt[3]{x}}$

(b) Which one of the following best describes the line tangent to the graph of $g(x)$ at the point $(0, 5)$?

i. The tangent line has slope 5.
ii. The tangent line does not exist.
iii. The tangent line is horizontal.
iv. The tangent line is vertical.



5. (10 points) The graphs of functions f and g are shown below. Use the graphs to estimate the function values required for each part of this problem.



(a) Determine $g'(4)$.

$g'(4) = 0$ (TANGENT LINE IS HORIZONTAL.)

(b) Explain why $f'(3)$ does not exist.

SHARP POINT $\Rightarrow$ SLOPE FROM LEFT $\neq$ SLOPE FROM RIGHT

(c) Let $h(x) = xf(x)$. Determine $h'(2)$.

$h'(x) = f(x) + x f'(x)$ $\quad$ $h'(2) = f(2) + 2f'(2) = 2 + 2(-1) = \boxed{0}$

(d) Let $h(x) = \frac{g(x)}{f(x)}$. Determine $h'(1)$.

$$h'(x) = \frac{f(x)g'(x) - g(x)f'(x)}{(f(x))^2} \qquad h'(1) = \frac{f(1)g'(1) - g(1)f'(1)}{[f(1)]^2} = \frac{3(1) - (1)(-1)}{9} = \frac{4}{9}$$

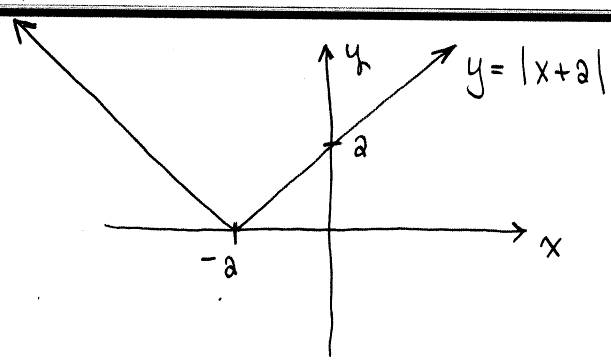
6. (5 points) An object is moving along a line in such a way that its position at any time t is given by $s(t) = t^3 - 5t^2 + 8t - 9$. At what time is the object's acceleration equal to zero?

$s'(t) = 3t^2 - 10t + 8$ $\quad$ $s''(t) = 0 \Rightarrow t = \frac{10}{6} = \frac{5}{3}$

$s''(t) = 6t - 10$ $\quad$ $t = \frac{5}{3}$

7. (4 points) Let $F(x) = |x+2|$.

(a) Sketch the graph of $F(x)$.



(b) Which one of the following best describes the line tangent to the graph of $F(x)$ at the point $(-2, 0)$?

i. The tangent line is horizontal.

ii. The tangent line is vertical.

iii. The tangent line does not exist. SHARP PT AT $X = -2$

iv. The tangent line has slope 1.

8. (20 points) Differentiate. Do not simplify.

(a) $\frac{d}{dx}\left(x^{5/7} + 2x^{-1} - \frac{1}{x^2}\right) = \frac{d}{dx}\left(x^{5/7} + 2x^{-1} - x^{-2}\right) = \boxed{\frac{5}{7}x^{-2/7} - 2x^{-2} + 2x^{-3}}$

(b) $\frac{d}{dt}\left(\frac{t \sin t}{\cos t}\right) = \frac{d}{dt}\left(t \tan t\right) = \boxed{\tan t + t \sec^2 t}$

(c) $\frac{d}{dx}\tan(\pi x^2) = \sec^2(\pi x^2)\,\frac{d}{dx}(\pi x^2)$

$= \boxed{2\pi x \sec^2(\pi x^2)}$

(d) $\frac{d}{dt}\left[t^2 (9t-4)^8\right] = 2t(9t-4)^8 + t^2(8)(9t-4)^7(9)$

$= \boxed{2t(9t-4)^8 + 72t^2(9t-4)^7}$

9. An object is launched vertically so that its height (in feet) after t seconds is given by

$$s(t) = -16t^2 + 48t + 160.$$

Include units with your answer for each part of this problem.

(a) (3 points) Determine the average rate of change the object's height over the interval from $t = 1$ to $t = 3$.

$$\frac{\Delta s}{\Delta t} = \frac{s(3) - s(1)}{3-1} = \frac{160 - 192}{3-1} = \frac{-32}{2} = -16 \text{ FT/SEC}$$

(b) (3 points) Determine the object's velocity at time $t = 3$.

$$v(t) = s'(t) = -32t + 48$$

$$v(3) = -96 + 48 = -48 \text{ FT/SEC}$$

(c) (2 points) What is the acceleration of the object?

$$a(t) = v'(t) = -32 \text{ FT/SEC}^2$$

(d) (3 points) When does the object hit the ground?

$$s(t) = 0 \Rightarrow -16t^2 + 48t + 160 = 0$$

$$-16(t^2 - 3t - 10) = 0 \qquad -16(t-5)(t+2) = 0$$

$$t = 5 \text{ SEC}$$

(e) (2 points) What is the object's speed when it hits the ground?

$$|s'(5)| = |-32(5) + 48| = 112 \text{ FT/SEC}$$

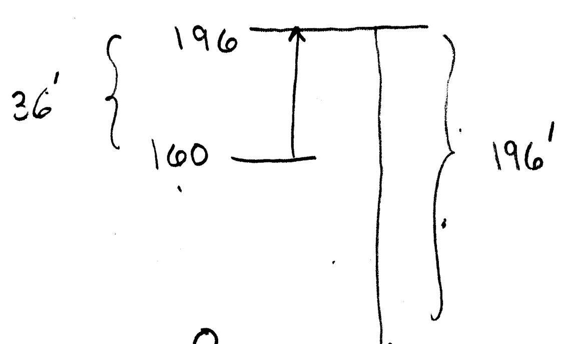
(f) (4 points) Determine the object's maximum height.

$$s'(t) = -32t + 48 = 0$$

$$\Rightarrow t = \frac{48}{32} = 1.5 \text{ SEC}$$

$$s(1.5) = 196 \text{ FT}$$

(g) (2 points) What is the overall length of the object's path?



$$196 + 36 = 232 \text{ FT}$$

10. (12 points) The graph of the equation $x^3 + xy^2 = 3x^2 - y^2$ is called a *trisectrix of Maclaurin.*

(a) Use implicit differentiation to find a formula for dy/dx.

$$\frac{d}{dx}\left(x^3 + xy^2\right) = \frac{d}{dx}\left(3x^2 - y^2\right)$$

$$3x^2 + y^2 + 2xy\frac{dy}{dx} = 6x - 2y\frac{dy}{dx}$$

$$2xy\frac{dy}{dx} + 2y\frac{dy}{dx} = 6x - 3x^2 - y^2$$

$$\boxed{\frac{dy}{dx} = \frac{6x - 3x^2 - y^2}{2xy + 2y}}$$

(b) Use dy/dx to compute the slope of the graph at the point $(1, -1)$. Then determine an equation for the tangent line at $(1, -1)$.

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(1,-1)} = \frac{6-3-1}{-2+-2} = \frac{2}{-4} = -\frac{1}{2}$$

$$\boxed{y + 1 = -\frac{1}{2}(x-1) \quad \text{OR} \quad y = -\frac{1}{2}x - \frac{1}{2}}$$

11. (6 points) Briefly describe three ways that a derivative can fail to exist. (Hint: At least two ways have already come up on this test.)

THE DERIVATIVE DOES NOT EXIST WHEN

① THE FUNCTION IS DISCONTINUOUS,

② THE GRAPH HAS A SHARP POINT, OR

③ THE GRAPH'S TANGENT LINE IS VERTICAL.