Math 131 - Assignment 10

April 17, 2024

Name Key

Score

Show all work to receive full credit. Supply explanations when necessary. Use extra paper as necessary. This assignment is due April 24.

1. Find the critical numbers of $f(x) = x^4 + 4x^3 - 36x^2$. Then use the 2nd derivative to determine if each gives relative max and relative min.

$f'(x) = 4x^3 + 12x^2 - 72x$

$= 4x(x+6)(x-3) = 0$

$\underbrace{x=0,\ x=-6,\ x=3}_{\text{ALL CRIT \#'s}}$

$f''(x) = 12x^2 + 24x - 72$

$f''(0) = -72 \Rightarrow f(0) = 0$ IS A REL MAX

$f''(-6) = 216 \Rightarrow f(-6) = -864$ IS A REL MIN

$f''(3) = 108 \Rightarrow f(3) = 135$ IS A REL MIN

2. Find the limit, showing all work. Do not use L'Hôpital's rule.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)\left(\frac{x^2+1}{x^2-1}\right)$$

∞/∞

MULT BY $\frac{1/x^2}{1/x^2}$

$$= \lim_{x\to\infty}\left(1+\frac{1}{x}\right)\cdot\lim_{x\to\infty}\left(\frac{x^2+1}{x^2-1}\right)$$

$$= (1)\cdot\lim_{x\to\infty}\left(\frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}\right) = (1)\cdot(1) = \boxed{1}$$

3. Evaluate the limit: $\underbrace{\lim_{x\to-\infty}\frac{\sqrt{4x^2-1}}{x+2}\cdot\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}}}$

IN THIS CASE,

$\sqrt{x^2} = |x| = -x$,

SO THAT $\frac{1}{\sqrt{x^2}} = -\frac{1}{x}$

$$\lim_{x\to-\infty}\frac{\sqrt{4-\frac{1}{x^2}}}{-1-\frac{2}{x}} = \frac{\sqrt{4}}{-1} = -2$$

4. Find the horizontal and vertical asymptotes of the graph of $h(x) = \dfrac{2-x^2}{x^2+x}$. Show work or explain your reasoning.

H.A...

$$\lim_{x\to\pm\infty} \frac{2-x^2}{x^2+x} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\pm\infty} \frac{\frac{2}{x^2}-1}{1+\frac{1}{x}} = -1$$

H.A. IS $y=-1$

V.A...

$$h(x) = \frac{2-x^2}{x(x+1)}$$

$X=0$ AND $X=-1$ ARE ASSOCIATED WITH $\frac{k\neq 0}{0}$ FORMS

$\boxed{\text{V.A. } X=0,\ X=-1}$

5. Find the vertical and horizontal asymptotes of the graph of $f(x) = \dfrac{x \sin x}{x^2-1}$. Show work or explain your reasoning.

H.A...

$$\lim_{x\to\pm\infty} \frac{x \sin x}{x^2-1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\pm\infty} \frac{\frac{\sin x}{x}}{1-\frac{1}{x^2}} = \frac{0}{1} = 0$$

H.A. IS $y=0$

V.A...

$$f(x) = \frac{x \sin x}{(x-1)(x+1)}$$

$X=1,\ X=-1$ ARE ASSOCIATED WITH $\frac{k\neq 0}{0}$ FORMS.

$\boxed{\text{V.A. } X=1,\ X=-1}$

6. Use L'Hôpital's rule to find each limit.

(a) $\displaystyle\lim_{x\to 0} \frac{\arctan x}{\sin x}$ $0/0$

$$= \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{\cos x} = \frac{\frac{1}{1}}{1} = \boxed{1}$$

(b) $\displaystyle\lim_{x\to\infty} \frac{x^3}{e^{x^2}}$ ∞/∞

$$= \lim_{x\to\infty} \frac{3x^2}{2x e^{x^2}} \quad \infty/\infty$$

$$= \lim_{x\to\infty} \frac{6x}{2e^{x^2}+4x^2e^{x^2}} \quad \infty/\infty$$

$$= \lim_{x\to\infty} \frac{6}{4xe^{x^2}+8xe^{x^2}+8x^3e^{x^2}} = 6/\infty = \boxed{0}$$

$\infty \cdot 0$

7. Evaluate the limit: $\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right)$

$0/0$

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}}$$

$$= \cos(0) = \boxed{1}$$

8. Try using L'Hôpital's rule to compute $\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$. What happens? Can you determine the limit by using techniques we learned earlier?

L'Hopital's Rule gives

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \ldots$$

It fails to be helpful!

Mult by $\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}}$.

In this case, $\frac{1}{\sqrt{x^2}} = \frac{1}{x}$

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} \cdot \frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{x}} =$$

$$\lim_{x\to\infty} \sqrt{1+\frac{1}{x^2}} = \sqrt{1} = \boxed{1}$$

9. Find $f(x)$ if $f'(x) = \frac{2}{x^2} - \frac{x^2}{2}$ and $f(1) = 0$.

$$f'(x) = 2x^{-2} - \tfrac{1}{2}x^2$$

$$f(x) = -2x^{-1} - \tfrac{1}{6}x^3 + C$$

$$= -\frac{2}{x} - \frac{x^3}{6} + C$$

$$f(1) = 0 \Rightarrow -2 - \tfrac{1}{6} + C = 0$$

$$\Rightarrow C = \frac{13}{6}$$

$$\boxed{f(x) = -\frac{2}{x} - \frac{x^3}{6} + \frac{13}{6}}$$

10. Let $f(x) = 6x^2 - \sec x \tan x$. Determine the antiderivative of f whose graph passes through the point $(0, 5)$.

$$g(x) = \int (6x^2 - \sec x \tan x)\,dx$$

$$g(x) = 2x^3 - \sec x + C$$

$$\boxed{g(x) = 2x^3 - \sec x + 6}$$

$$g(0) = 5 \Rightarrow -\sec(0) + C = 5$$

$$C = 6$$