Math 131 - Assignment 6

February 28, 2024

Name Key

Score ______

Show all work to receive full credit. Supply explanations when necessary. Use extra paper as necessary. This assignment is due March 6.

1. Determine each derivative.

   (a) $\frac{d}{dx}\sin^2(x^2) = \frac{d}{dx}\left[\sin(x^2)\right]^2 = 2\sin(x^2)\,\frac{d}{dx}\sin(x^2)$

   $= 2\sin(x^2)\cos(x^2)(2x)$

   $= \boxed{4x\sin(x^2)\cos(x^2)}$

   (b) $\frac{d}{dx}\left(\frac{x}{\sqrt{x^4+4}}\right) = \frac{\left(\sqrt{x^4+4}\right)(1) - (x)\left(\frac{1}{2}\right)\left(x^4+4\right)^{-1/2}\left(4x^3\right)}{\left(\sqrt{x^4+4}\right)^2}$

   $= \boxed{\frac{(x^4+4)^{1/2} - 2x^4(x^4+4)^{-1/2}}{x^4+4}}$

2. Find all points on the graph of $y = \sqrt[3]{(x^2-1)^2}$ at which $dy/dx = 0$ or dy/dx is not defined.

   $(x^2-1)^2 = x^4 - 2x^2 + 1$

$y = \sqrt[3]{x^4-2x^2+1} = \left(x^4-2x^2+1\right)^{1/3}$

$\frac{dy}{dx} = \frac{1}{3}\left(x^4-2x^2+1\right)^{-2/3}\left(4x^3-4x\right)$

$= \frac{4x(x^2-1)}{3\left[(x^2-1)^2\right]^{2/3}} = \frac{4x(x^2-1)}{3(x^2-1)^{4/3}}$

$= \frac{4x}{3(x^2-1)^{1/3}} \longrightarrow$

$\frac{dy}{dx} = 0 \Rightarrow 4x = 0 \Rightarrow \boxed{x=0,\ y=1}$

$\frac{dy}{dx}$ DNE $\Rightarrow x^2-1=0 \Rightarrow \boxed{x=\pm 1,\ y=0}$

3. You are given the following information:

$$g(5) = -3, \quad g'(5) = 6, \quad h(5) = 3, \quad h'(5) = -2.$$

For each part below, use the information to determine $f'(5)$. If it is not possible to do so, say what additional information would be required.

(a) $f(x) = g(x)h(x)$

$$f'(x) = g'(x)h(x) + g(x)h'(x) \Rightarrow f'(5) = (6)(3) + (-3)(-2) = \boxed{24}$$

(b) $f(x) = \dfrac{g(x)}{h(x)}$

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2} \Rightarrow f'(5) = \frac{(3)(6) - (-3)(-2)}{(3)^2} = \frac{12}{9} = \boxed{\frac{4}{3}}$$

(c) $f(x) = g(h(x))$

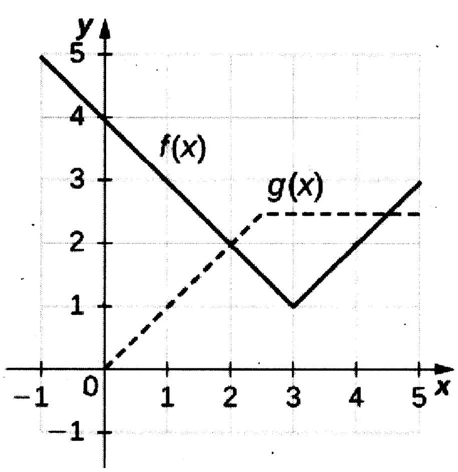
$$f'(x) = g'(h(x))h'(x) \Rightarrow f'(5) = g'(3)(-2) = -2g'(3)$$

NEED THIS VALUE!

(d) $f(x) = [g(x)]^3$

$$f'(x) = 3[g(x)]^2 g'(x) \Rightarrow f'(5) = 3[-3]^2(6) = 3 \cdot 9 \cdot 6 = \boxed{162}$$

4. The graphs of f and g are shown below. Use the chain rule and information from the graphs to determine the derivative of $g(f(x))$ when $x = 1$.



DERIV OF $g(f(x)) =$

$g'(f(x))f'(x)$.

WHEN $x = 1$,

$g'(f(1))f'(1)$

$= g'(3)(-1)$

$= (0)(-1) = \boxed{0}$

5. Given the equation $x^3 + 8xy + y^3 = 25x$, use implicit differentiation to determine $\frac{dy}{dx}$ at the point $(x, y) = (1, 2)$.

$$\frac{d}{dx}\left(x^3 + 8xy + y^3\right) = \frac{d}{dx}(25x)$$

$$3x^2 + 8y + 8x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 25$$

$$\left(8x + 3y^2\right)\frac{dy}{dx} = 25 - 3x^2 - 8y$$

$$\boxed{\frac{dy}{dx} = \frac{25 - 3x^2 - 8y}{8x + 3y^2}}$$

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,2)} = \frac{25 - 3 - 16}{8 + 12} = \frac{6}{20} = \frac{3}{10}$$

6. Given the equation $y^3 + y^2 - 5y - x^2 = -4$, use implicit differentiation to determine $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(y^3 + y^2 - 5y - x^2\right) = 0$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

$$\left(3y^2 + 2y - 5\right)\frac{dy}{dx} = 2x$$

$$\boxed{\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}}$$

7. Find equations for the tangent line and normal line at the point $(2, 3)$.

$$x^3 + y^3 = 6xy - 1$$

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(6xy - 1)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

$$3x^2 - 6y = \left(6x - 3y^2\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3x^2 - 6y}{6x - 3y^2}$$

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(2,3)}$$

$$= \frac{12 - 18}{12 - 27} = \frac{-6}{-15} = \frac{2}{5}$$

TANGENT LINE:

$m = \frac{2}{5}$, POINT $(2,3)$

$$\boxed{y - 3 = \frac{2}{5}(x - 2) \quad \text{OR} \quad y = \frac{2}{5}x + \frac{11}{5}}$$

NORMAL LINE: $m = -\frac{5}{2}$, POINT $(2,3)$

$$\boxed{y - 3 = -\frac{5}{2}(x - 2) \quad \text{OR} \quad y = -\frac{5}{2}x + 8}$$

8. Let $f(x) = x^5 + 7x - 9$.

(a) Compute $f^{-1}(-1)$.

$$f^{-1}(-1) = u \Leftrightarrow u^5 + 7u - 9 = -1 \Leftrightarrow u = 1$$

$$\boxed{f^{-1}(-1) = 1}$$

(b) Compute $(f^{-1})'(-1)$.

$$\frac{1}{f'(f^{-1}(-1))} = \frac{1}{f'(1)} = \boxed{\frac{1}{12}}$$

$$f'(x) = 5x^4 + 7$$

$$f'(1) = 12$$

(c) Compute $f^{-1}(11)$. (You'll probably have to use a calculator to approximate the value.)

$$f^{-1}(11) = u \Leftrightarrow u^5 + 7u - 9 = 11 \Leftrightarrow u^5 + 7u - 20 = 0$$

$$\Leftrightarrow u \approx 1.5556386$$

$$\boxed{f^{-1}(11) \approx 1.5556386}$$

(d) Compute $(f^{-1})'(11)$.

$$\frac{1}{f'(f^{-1}(11))} = \frac{1}{f'(1.5556386)}$$

$$f'(x) = 5x^4 + 7$$

$$f'(1.5556386) \approx 36.28228$$

$$\approx \frac{1}{36.28228} \approx \boxed{0.02756}$$