Math 131 - Test 1

February 7, 2024

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. Determine all limits analytically unless otherwise indicated. When evaluating limits, you may need to use $+\infty$, $-\infty$, or DNE (does not exist). When classifying discontinuities, use the words *removable*, *nonremovable*, *jump*, and/or *infinite*.

1. (6 points) Estimate the following limit by using a table of appropriate numerical values. Use a sufficient number of values to feel confident about your result when rounded to three decimal places.

$$\lim_{x\to 0^+}(1+x)^{1/x} \approx \boxed{2.718}$$

X	$(1+X)^{1/X}$
0.1	2.5937
0.01	2.7048
0.001	2.7169
0.0001	2.7181
0.00001	2.7183
0.000001	2.7183

2. (9 points) These limits DO NOT EXIST. Carefully explain why each limit fails to exist.

(a) $\lim_{x\to\pi/2}\frac{\tan x}{x}$

THE GRAPH OF $y = \tan x$ HAS A VERTICAL ASYMPTOTE AT $X = \frac{\pi}{2}$ (WE KNOW FROM TRIG!)

$\boxed{\frac{\tan X}{X} \text{ GROWS WITHOUT BOUND AS } X \to \pi/2}$

(b) $\lim_{x\to 0}\frac{2x^3+x}{|x|}$ $0/0$

$\lim_{X\to 0^+}\frac{X(2X^2+1)}{X} = 1$, $\lim_{X\to 0^-}\frac{X(2X^2+1)}{-X} = -1$

$\boxed{\text{LIMIT FROM RIGHT} \neq \text{LIMIT FROM LEFT}}$

(c) $\lim_{x\to 3}\sqrt{9-x^2}$

IT'S A TWO-SIDED LIMIT, BUT $\sqrt{9-X^2}$ IS NOT DEFINED TO THE RIGHT OF $X=3$.

3. (24 points) Determine each limit analytically, or explain why the limit does not exist.

(a) $\lim_{x \to 20} \frac{x - 20}{\sqrt{x-4} - 4}$ 0/0 More work

$$\lim_{x \to 20} \frac{x-20}{\sqrt{x-4}-4} \cdot \frac{\sqrt{x-4}+4}{\sqrt{x-4}+4} = \lim_{x \to 2} \frac{(x-20)\left[\sqrt{x-4}+4\right]}{x-4-16}$$

$$= \lim_{x \to 20} \frac{\cancel{(x-20)}\left[\sqrt{x-4}+4\right]}{\cancel{x-20}} = \sqrt{16}+4 = \boxed{8}$$

(b) $\lim_{w \to -5} \frac{2w^2 + 10w}{w^2 - w - 30}$ 0/0 More work

$$\lim_{w \to -5} \frac{\cancel{(w+5)}(2w)}{\cancel{(w+5)}(w-6)} = \frac{2(-5)}{-5-6} = \frac{-10}{-11} = \boxed{\frac{10}{11}}$$

(c) $\lim_{x \to 2} \frac{\frac{1}{2} - \frac{1}{x}}{x - 2}$ 0/0 More work.

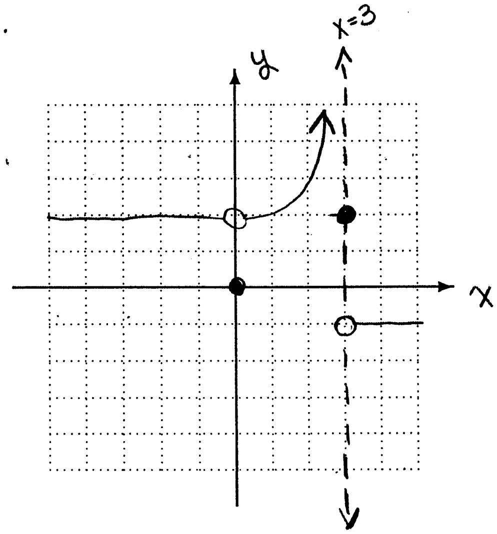
$$\lim_{x \to 2} \frac{\frac{x}{2x} - \frac{2}{2x}}{x-2} = \lim_{x \to 2} \frac{\frac{x-2}{2x}}{x-2} = \lim_{x \to 2} \frac{\cancel{x-2}}{2x\cancel{(x-2)}} = \boxed{\frac{1}{4}}$$

(d) $\lim_{r \to -2} \frac{r^2 + 6r + 8}{r^2 + 4} = \frac{4-12+8}{(-2)^2+4} = \frac{0}{8} = \boxed{0}$

Direct substitution.

4. (7 points) Sketch the graph of a function f such that

- f is defined for all real numbers between -5 and 5,
- $\lim_{x \to 0} f(x) = 2$,
- $f(0) = 0$,
- $\lim_{x \to 3^-} f(x) = \infty$,
- $\lim_{x \to 3^+} f(x) = -1$, and
- $f(3) = 2$.



5. (10 points) Consider the function $f(x) = \begin{cases} x^2 + x + 2, & x < 0 \\ \dfrac{\sin 4x}{2x}, & x > 0 \end{cases}$

(a) Evaluate the limit: $\lim_{x \to \pi} f(x)$. $= \dfrac{\sin 4\pi}{2\pi} = \dfrac{0}{2\pi} = \boxed{0}$

(b) Evaluate the limit: $\lim_{x \to 0+} f(x)$. $\lim_{x \to 0} \dfrac{\sin 4x}{2x} = \lim_{x \to 0} \dfrac{2 \sin 4x}{4x} = 2(1) = \boxed{2}$

(c) Evaluate the limit: $\lim_{x \to 0-} f(x)$. $\lim_{x \to 0} (x^2 + x + 2) = \boxed{2}$

(d) Is f continuous at $x = 0$? Explain why or why not.

No. THE LIMIT AT $X = 0$ EXISTS, BUT $f(0)$ IS NOT DEFINED.

(e) If your answer to part (d) was "no," then classify the discontinuity. Otherwise, draw a smiley face.

REMOVABLE. (LIMIT AT $X = 0$ EXISTS.)

6. (4 points) Give an example of a function with a removable discontinuity at $x = 3$ and an infinite discontinuity at $x = 1$.

$\boxed{R(x) = \dfrac{X-3}{(X-1)(X-3)}}$

REASONS AT $X=3$, THE LIMIT EXISTS AND IS $\frac{1}{2}$.

$X=1$ IS ASSOCIATED WITH A $\frac{\text{NONZERO}}{\text{ZERO}}$ FORM.

7. (12 points) In each part below, determine analytically whether the limit is $+\infty$, $-\infty$, or DNE. Show work or explain your reasoning.

ALL ARE $1/0$ FORMS - SOME KIND OF UNBOUNDED GROWTH

(a) $\lim_{x \to -4} \dfrac{x+5}{(x+4)^2}$

NEAR $X=-4$ (BOTH SIDES), NUMERATOR IS POSITIVE $(\approx +1)$ AND DENOMINATOR IS POSITIVE (A SQUARE).

LIMIT IS $\boxed{+\infty}$

(b) $\lim_{x \to -4^-} \dfrac{x+5}{x+4}$

LEFT OF $X=-4$: $\dfrac{X+5}{X+4} = \dfrac{+}{-} \Rightarrow$ LIMIT IS $\boxed{-\infty}$

(c) $\lim_{x \to -4^+} \dfrac{x+5}{x+4}$

RIGHT OF $X=-4$: $\dfrac{X+5}{X+4} = \dfrac{+}{+} \Rightarrow$ LIMIT IS $\boxed{+\infty}$

(d) $\lim_{x \to -4} \dfrac{x+5}{x+4}$ $\boxed{\text{DNE}}$ BECAUSE OF (b) & (c)

8. (4 points) Use the limit laws to rewrite the limit in terms of only limits of x and limits of constants. Then give the value of the limit.

$$\lim_{x \to 1}(x^2 + 5x - 8)$$

$$\left(\lim_{X\to 1} X\right)^2 + \left(\lim_{X\to 1} 5\right)\left(\lim_{X\to 1} X\right) - \left(\lim_{X\to 1} 8\right)$$

$$= (1)^2 + (5)(1) - (8) = \boxed{-2}$$

9. (4 points) Suppose $\lim_{x\to 7} f(x) = 2$, $\lim_{x\to 7} h(x) = 2$, and $f(x) \le g(x) \le h(x)$ for all x. What can you say about $\lim_{x\to 7} g(x)$? Explain your reasoning.

$f(x) \le g(x) \le h(x)$ AND $\lim_{x\to 7} f(x) = \lim_{x\to 7} h(x) = 2$

$\Rightarrow$ $\boxed{\lim_{x\to 7} g(x) = 2 \text{ BY SQUEEZE THM.}}$

10. (5 points) Show that g is not continuous at $x = 5$.

$$g(x) = \begin{cases} \sqrt{x^2+2x+1}, & x < 5 \\ 6, & x = 5 \\ x + 1 - \cos \pi x, & x > 5 \end{cases}$$

$g(5) = 6$

$\lim_{x\to 5^-} g(x) = \lim_{x\to 5} \sqrt{x^2+2x+1} = \sqrt{36} = 6$

$\lim_{x\to 5^+} g(x) = \lim_{x\to 5} (x+1-\cos(\pi x)) = 5+1-\cos(5\pi) = 7$

} LIMIT AT $x=5$ DNE!

JUMP DISCONT!

11. (5 points) Indicate whether each statement is true (T) or false (F).

(a) F A jump discontinuity might also be a removable discontinuity.

LIMIT MUST EXIST FOR A REMOVABLE DISCONT.

(b) F $\lim_{x\to 0} \sqrt{x} = 0$ $\lim_{x\to 0^-} \sqrt{x}$ DNE

(c) T The limit of a polynomial function can always be found by direct substitution.

(d) T If $\lim_{x\to 2} f(x) = f(2)$, then f is continuous at $x = 2$. DEFINITION OF CONTINUITY AT $x=2$.

(e) F The limit of a rational function can always be found by direct substitution.

NOT IF YOU GET DIVISION BY ZERO.

SUCH AS $\lim_{x\to 2} \frac{x^2-4}{x-2}$

12. (2 points) Suppose you were asked to use a table of values to estimate $\lim_{x \to 1} f(x)$. Which list of x-values shown below would be best for your table?

(a) $x = 0.9, 0.99, 0.999, 1.1, 1.11, 1.111$ ← 1.11, 1.111 NOT GETTING CLOSE TO 1.
(b) $x = 0.9, 0.99, 0.999, 1.0, 1.1, 1.01, 1.001$ ← DON'T WANT 1
(c) $x = 0.9, 0.99, 0.999, 1.1, 1.01, 1.001$
(d) $x = 1.00001, 1.000001, 1.0000001, 1.00000001, 1.000000001$ ← ONE-SIDE

13. (2 points) Which one of the following best describes the meaning of the statement $\lim_{x \to 0} g(x) = \infty$?

(a) Direct substitution results in division by zero.
(b) The limit at $x = 0$ exists, and it is a very large positive number.
(c) The limit at $x = 0$ does not exist because $g(0)$ is not defined.
(d) The limit at $x = 0$ does not exist because the values of g grow positively without bound as $x \to 0$.

14. (2 points) Which of these IS a reason that a limit may not exist?

(a) The function is not defined to the right of the limit point.
(b) The function is not defined at the limit point.
(c) Direct substitution cannot be applied.
(d) The function is not continuous at the limit point.

15. (2 points) Suppose $\lim_{x \to c} f(x) = \infty$. Which one of the following is NOT necessarily true?

(a) The graph of f has a vertical asymptote at $x = c$.
(b) $\lim_{x \to c^+} f(x) = \infty$
(c) $\lim_{x \to c^-} f(x) = \infty$
(d) f is not defined at $x = c$.

16. (2 points) Suppose $\lim_{x \to 3} f(x) = 10$. Which one of these statements must be true?

(a) f is continuous at $x = 3$.
(b) f is defined at $x = 3$.
(c) $f(3) = 10$
(d) $\lim_{x \to 3^+} f(x) = 10$