**Math 131 - Test 2**
March 6, 2024

Name Key

Score

**Show all work to receive full credit.** Supply explanations where necessary. Unless otherwise indicated, use differentiation rules for all derivatives and do not simplify.

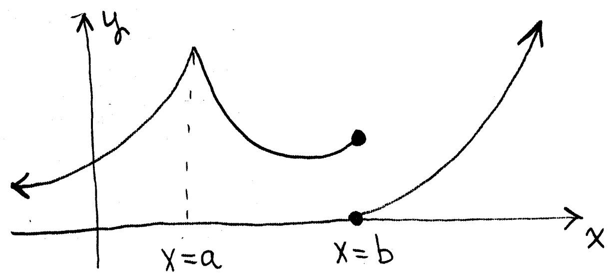
1. (8 points) We studied three ways in which a derivative can fail to exist. Describe them. Then sketch the graph of a function that illustrates exactly two of the three failures and indicate exactly where those failures occur.

f IS NOT DIFFERENTIABLE

① AT ANY POINT OF DISCONTINUITY,

② AT ANY POINT WHERE SLOPE FROM LEFT IS NOT EQUAL TO SLOPE FROM RIGHT (i.e., GRAPH HAS A SHARP PT),

③ AT ANY POINT WHERE THE GRAPH HAS A VERTICAL TANGENT LINE.




2. (5 points) Suppose $f$ is a function for which $f'(5)$ exists. Which of the following are equal to $f'(5)$? Circle all that apply.

(a) The slope of the line tangent to the graph of $f$ at $x = 5$

(b) The instantaneous rate of change of $f$ at $x = 5$

(c) $\lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$

(d) The slope of the secant line passing through the point where $x = 5$ $\quad \frac{f(5+h)-f(5)}{h}$ No LIMIT

(e) $\lim_{h \to 0} \frac{f(5-h) - f(5)}{h} = -f'(5)$

3. (10 points) Use the **limit definition of derivative** to find $f'(x)$ when $f(x) = x - 4x^2$. Once you have found $f'(x)$, use it to compute $f'(5)$.

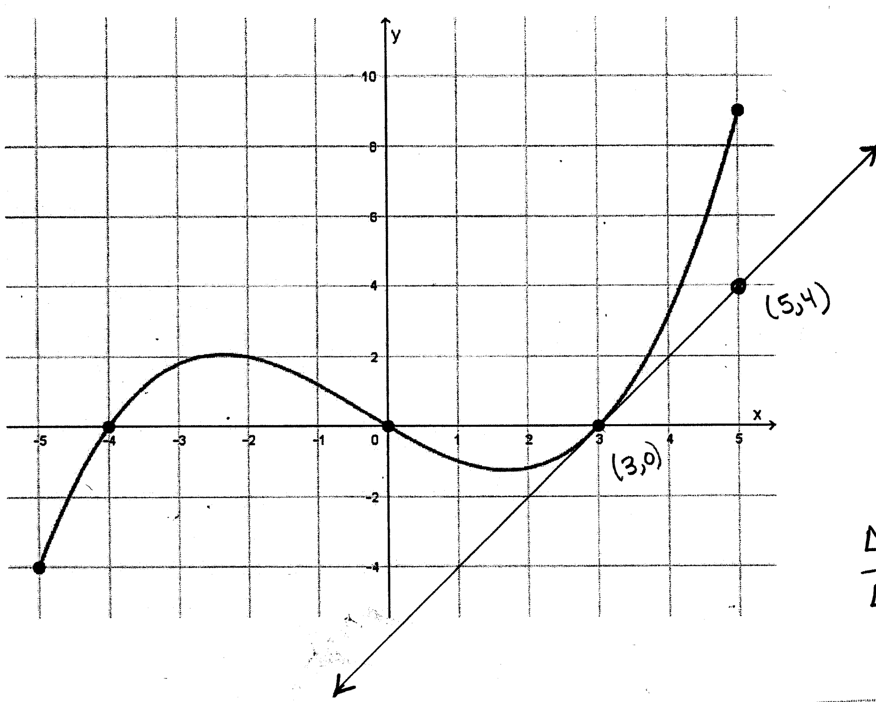
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left[(x+h)-4(x+h)^2\right] - \left[x-4x^2\right]}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h-\cancel{4x^2}-8xh-4h^2-\cancel{x}+\cancel{4x^2}}{h} = \lim_{h\to 0} \frac{\cancel{h}\cdot(1-8x-4h)}{\cancel{h}}$$

$$= \lim_{h\to 0} (1-8x-4h) = \boxed{1-8x}$$

$$f'(5) = 1-8(5) = \boxed{-39}$$

4. (5 points) The graph of $f$ is shown below. Use it to estimate the value of $f'(3)$. Briefly explain your reasoning.




Approximate tangent line at $X=3$

Its slope is

$$\frac{\Delta y}{\Delta x} = \frac{4-0}{5-3} = \frac{4}{2} = 2$$

$$\boxed{f'(3) \approx 2}$$

5. (6 points) Use the quotient rule and trig identities to derive our formula for the derivative of $y = \cot x$.

$$\frac{d}{dx}\cot x = \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{(\sin x)(-\sin x)-(\cos x)(\cos x)}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x \quad ✓$$

6. (20 points) Determine the derivative of each function. Do not simplify.

(a) $y = 5x^6 - x + 10 + \sqrt[5]{x^3} - \dfrac{3}{x^4}$ $= 5x^6 - x + 10 + x^{3/5} - 3x^{-4}$

$$\boxed{\frac{dy}{dx} = 30x^5 - 1 + \frac{3}{5}x^{-2/5} + 12x^{-5}}$$

(b) $f(x) = \cos(\pi x^2 + 1)$

$$f'(x) = -\sin(\pi x^2 + 1)\,(2\pi x) = \boxed{-2\pi x \sin(\pi x^2 + 1)}$$

(c) $g(t) = t^2 \left(\dfrac{2}{t^2} - \dfrac{1}{t^2+t+1}\right)$ $= 2 - \dfrac{t^2}{t^2+t+1}$

$$\boxed{g'(t) = -\left[\frac{2t(t^2+t+1) - t^2(2t+1)}{(t^2+t+1)^2}\right] = -\frac{t^2+2t}{(t^2+t+1)^2}}$$

(d) $f(x) = (\tan x + \sin 2x)^5$

$$\boxed{f'(x) = 5(\tan x + \sin 2x)^4(\sec^2 x + 2\cos 2x)}$$

7. (4 points) Suppose $M(t)$ is a differentiable function that gives the mass of a snowball, in kilograms, after $t$ minutes as it grows while rolling down a hill.

(a) What are the units on $M'(t)$?

$\frac{dM}{dt}$ HAS UNITS $\boxed{\text{kg}/\text{min}}$

(b) What are the units on $M''(t)$?

$\frac{d}{dt}\left(\frac{dM}{dt}\right)$. HAS UNITS $\boxed{\text{kg}/\text{min}^2}$

8. (14 points) On Mars, the acceleration due to the force of gravity is about $g = 3.7\,\text{m/s}^2$ (much less than on Earth). In an experiment on the surface of Mars, a probe throws an object straight upward so that its height in meters after $t$ seconds is given by

$$s(t) = -1.85t^2 + 1.3875t + 0.4625.$$

**Include units with your answer for each part of this problem.** Also, it may be helpful to know that $s(t)$ can be written $s(t) = -0.4625(4t^2 - 3t - 1)$.

(a) Determine the average velocity of the object over the first half second of travel.

$$\frac{\Delta s}{\Delta t} = \frac{s(0.5) - s(0)}{0.5} = \frac{0.69375 - 0.4625}{0.5} = \boxed{0.4625\ \text{m/s}}$$

(b) When will the object reach its maximum height?

$$s'(t) = -3.7t + 1.3875 \qquad s'(t) = 0 \Rightarrow t = \frac{1.3875}{3.7} = \boxed{0.375\,\text{s}}$$

(c) What is the maximum height of the object?

$$s(0.375) = 0.72265625\,\text{m} \approx \boxed{0.7227\,\text{m}}$$

(d) When does the object hit the ground?

$$s(t) = -0.4625(4t^2 - 3t - 1) = -0.4625(4t+1)(t-1) = 0$$

$$\Rightarrow \boxed{t = 1 \text{ SECOND}}$$

(e) What is the object's velocity when it hits the ground?

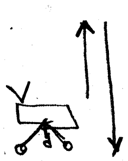
$$s'(1) = -3.7(1) + 1.3875 = \boxed{-2.3125\ \text{m/s}}$$

(f) What is the object's initial height?

$$\boxed{s(0) = 0.4625\,\text{m}}$$

(g) What is the overall length of the object's path?

2 × MAX HEIGHT − INITIAL HEIGHT



$$= \boxed{0.9828125\,\text{m} \approx 0.98\,\text{m}}$$

9. (12 points) Assume that $y$ is implicitly defined as a function of $x$ by the equation

$$x^3y^5 + 3x = 8y^3 + 6.$$

(a) Find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(x^3y^5 + 3x\right) = \frac{d}{dx}\left(8y^3 + 6\right)$$

$$3x^2y^5 + 5x^3y^4\frac{dy}{dx} + 3 = 24y^2\frac{dy}{dx}$$

$$5x^3y^4\frac{dy}{dx} - 24y^2\frac{dy}{dx} = -3x^2y^5 - 3$$

$$\boxed{\frac{dy}{dx} = \frac{-3x^2y^5 - 3}{5x^3y^4 - 24y^2}}$$

(b) Find an equation of the line tangent to the graph of the equation at the point $(2, 1)$.

$$m = \left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = \frac{-3(2)^2(1)^5 - 3}{5(2)^3(1)^4 - 24(1)^2} = \frac{-12-3}{40-24} = -\frac{15}{16}$$

$$\boxed{y - 1 = -\frac{15}{16}(x-2)}$$

(c) Find an equation of the line normal to the graph of the equation at the point $(2, 1)$.

$$m_{\perp} = \frac{16}{15}$$

$$\boxed{y - 1 = \frac{16}{15}(x-2)}$$

10. (8 points) Let $F(x) = x^2 \cos x$. Find $F''(x)$.

$F'(x) = 2x \cos x - x^2 \sin x$

$F''(x) = 2 \cos x - 2x \sin x - 2x \sin x - x^2 \cos x$

$$\boxed{F''(x) = 2 \cos x - 4x \sin x - x^2 \cos x}$$

11. (8 points) Determine the points (both coordinates) on the graph of $y = 2x^3 + 3x^2 - 36x$ at which the tangent line is horizontal. (You should find two such points.)

$\frac{dy}{dx} = 0$

$\frac{dy}{dx} = 6x^2 + 6x - 36$

$6x^2 + 6x - 36 = 0$

$6(x^2 + x - 6) = 0$

$6(x+3)(x-2) = 0$

$x = -3, \quad x = 2$

$\boxed{(x,y) = (-3, 81)}$ $\quad$ $\boxed{(x,y) = (2, -44)}$