# Math 131 - Test 3

April 10, 2024

Name key

Score

**Show all work to receive full credit.** Supply explanations where necessary.

1. (6 points) Let $f(x) = x^5 + x^3 - 30$. Find $f^{-1}(10)$ and then find $(f^{-1})'(10)$. (You must show work to get points.)

$(f^{-1})'(10) = \dfrac{1}{f'(f^{-1}(10))}$

   (a) $f^{-1}(10) = 2$ and $(f^{-1})'(10) = \dfrac{1}{92}$

   (b) $f^{-1}(10) = 100970$ and $(f^{-1})'(10) = \dfrac{1}{50300}$

   (c) $f^{-1}(10) = 30$ and $(f^{-1})'(10) = 50300$

   (d) $f^{-1}(10) = 2$ and $(f^{-1})'(10) = \dfrac{1}{40}$

$f^{-1}(10) = 2$ BECAUSE

$f(2) = 2^5 + 2^3 - 30 = 32 + 8 - 30 = 10$

AND

$\dfrac{1}{f'(2)} = \dfrac{1}{5(2)^4 + 3(2)^2} = \dfrac{1}{80+12} = \dfrac{1}{92}$

2. (4 points) Let $h(x) = \sin^{-1}(f(x))$. With the information below, compute $h'(3)$.

$$f(1) = \frac{1}{3}, \quad f'(1) = \frac{\sqrt{5}}{2}, \quad f(3) = \frac{\sqrt{3}}{2}, \quad f'(3) = \frac{1}{2}$$

$$h'(x) = \frac{1}{\sqrt{1 - [f(x)]^2}} \cdot f'(x)$$

$$h'(3) = \frac{1}{\sqrt{1 - [f(3)]^2}} f'(3) = \frac{1}{\sqrt{1 - \frac{3}{4}}} \cdot \frac{1}{2} = \frac{1}{1/2} \cdot \frac{1}{2} = \boxed{1}$$

3. (6 points) Determine each derivative.

   (a) $\dfrac{d}{dx}\left[x \cot^{-1}(x^2)\right] = \cot^{-1}(x^2) + x\left(\dfrac{-1}{1+x^4}\right)(2x) = \boxed{\cot^{-1}(x^2) - \dfrac{2x^2}{1+x^4}}$

   (b) $\dfrac{d}{dt}\left(\dfrac{5}{e^{\sqrt{t}}}\right) = \dfrac{d}{dt} 5e^{-\sqrt{t}} = \left(5e^{-\sqrt{t}}\right)\left(-\dfrac{1}{2\sqrt{t}}\right) = \boxed{-\dfrac{5}{2}\dfrac{e^{-\sqrt{t}}}{\sqrt{t}}}$

4. (5 points) Find the slope of the line tangent to the graph of $y = \log_8(x^3 + x)$ at the point where $x = 2$. Write your final answer in decimal form, rounded to the nearest thousandth.

$$y = \frac{\ln(x^3+x)}{\ln 8}$$

$$\frac{dy}{dx} = \frac{1}{\ln 8} \frac{3x^2+1}{x^3+x}$$

$x = 2$ MAKES

$$\frac{dy}{dx} = \boxed{\frac{13}{10 \ln 8} \approx 0.625}$$

5. (8 points) Use logarithmic differentiation to find $\frac{dy}{dx}$ when $y = \frac{2x^5}{\sqrt{x+1}(x^2+1)}$

$$\ln y = \ln 2 + 5 \ln x - \frac{1}{2} \ln(x+1) - \ln(x^2+1)$$

$$\frac{1}{y}\frac{dy}{dx} = 0 + \frac{5}{x} - \frac{1}{2}\frac{1}{x+1} - \frac{2x}{x^2+1}$$

$$\boxed{\frac{dy}{dx} = \left(\frac{5}{x} - \frac{1}{2x+2} - \frac{2x}{x^2+1}\right)\left(\frac{2x^5}{\sqrt{x+1}(x^2+1)}\right)}$$

6. (4 points) Find the instantaneous rate of change of $g(x) = 2^{3x+1}$ at the point where $x = 1$.

$$g'(x) = 2^{3x+1}(\ln 2)(3)$$

$$g'(1) = \boxed{2^4 \cdot \ln 2 \cdot 3 \approx 33.27}$$

7. (6 points) A particle is moving along the graph of $\underbrace{x^2 + y^3 = 3}$ in such a way that $\frac{dy}{dt} = -8$. Find $\frac{dx}{dt}$ when $x = 2$.

WHEN $x = 2$, $y = -1$

$$\frac{d}{dt}(x^2+y^3) = \frac{d}{dt}(3)$$

$$\underbrace{2x\frac{dx}{dt} + 3y^2\frac{dy}{dt} = 0}$$

$$2(2)\frac{dx}{dt} + 3(-1)^2(-8) = 0$$

$$\boxed{\frac{dx}{dt} = 6}$$

8. (6 points) A big block of ice is in the shape of a perfect cube. As it melts, the length of each edge of the cube is decreasing at a rate of 2 cm/hr. At what rate is the block's volume changing when the side length is 20 cm?

$V = s^3$, V= VOLUME AT TIME t, S= SIDE LENGTH AT TIME t

$\frac{ds}{dt} = -2 \ \text{cm}/\text{hr}$

FIND $\frac{dV}{dt}$ WHEN S= 20 cm

$\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$

WHEN S= 20...

$\frac{dV}{dt} = 3(20)^2(-2)$

$= \boxed{-2400 \ \text{cm}^3/\text{hr}}$

9. (5 points) Find the linearization of $f(x) = \tan^{-1} x$ at $x = 1$. Then use your linearization to approximate $f(0.92)$.

$f(1) = \tan^{-1}(1) = \frac{\pi}{4}$

$f'(x) = \frac{1}{1+x^2}$, $f'(1) = \frac{1}{2}$

$\boxed{L(x) = \frac{\pi}{4} + \frac{1}{2}(x-1)}$

$f(0.92) \approx L(0.92)$

$= \frac{\pi}{4} + \frac{1}{2}(-0.08)$

$= \frac{\pi}{4} - 0.04$

$\approx 0.7454$

10. (6 points) Let $y = e^{4x}\cos x$. Determine the differential $dy$. Then use differentials to estimate $\Delta y$ when $x$ changes from $x = 0$ to $x = 0.94$.

$\boxed{dy = (4e^{4x}\cos x - e^{4x}\sin x)\,dx}$

$\Delta y \approx (4e^0 \cos 0 - e^0 \sin 0)(0.94)$

$= (4)(0.94) = \boxed{3.76}$

11. (6 points) Explain what it means to be a critical number for a function $f$. Then say what you would look for on the graph of $f$ if you were trying to use the graph to identify critical numbers.

A CRITICAL NUMBER IS A DOMAIN INTERIOR PT AT WHICH $f'(x) = 0$ OR $f'(x)$ DNE.

LOOK AT THE GRAPH FOR FLAT SPOTS, SHARP POINTS, VERTICAL TAN. LINES, OR DISCONTINUITIES. AT DOMAIN INTERIOR POINTS.

12. (6 points) Let $h(x) = x\sqrt{2x+1}$. Find all $x$-values for which $h'(x) = 0$ or $h'(x)$ DNE. Then say which of those values are critical numbers.

$$h'(x) = (2x+1)^{1/2} + (x)\left(\tfrac{1}{2}\right)(2x+1)^{-1/2}(2) = \sqrt{2x+1} + \frac{x}{\sqrt{2x+1}}$$

$$= \frac{2x+1}{\sqrt{2x+1}} + \frac{x}{\sqrt{2x+1}} = \frac{3x+1}{\sqrt{2x+1}}$$

$h'(x) = 0 \Rightarrow 3x+1 = 0 \Rightarrow x = -1/3$

$h'(x)$ DNE $\Rightarrow 2x+1 = 0 \Rightarrow x = -1/2$

$X = -\frac{1}{2}$ IS NOT A CRIT # BECAUSE IT'S A DOMAIN BOUNDARY PT.

$X = -\frac{1}{3}$ IS A CRIT #.

13. (8 points) Use calculus techniques to find the absolute maximum and minimum values of $f(x) = 3x^4 + 2x^3 - 3x^2$ on $[-2, 1]$.

CRIT NUMBERS:

$$f'(x) = 12x^3 + 6x^2 - 6x$$
$$= 6x(2x^2 + x - 1)$$
$$= 6x(2x-1)(x+1) = 0$$

$x = 0,\ x = \frac{1}{2},\ x = -1$ (ALL ARE CRIT #'S)

DOMAIN ENDPTS: $x = -2,\ x = 1$

| $x$ | $f(x)$ | |
|---|---|---|
| 0 | 0 | |
| 1/2 | $-5/16$ | |
| $-1$ | $-2$ | ← ABS MIN |
| $-2$ | 20 | ← ABS MAX |
| 1 | 2 | |

14. (13 points) Let $f(x) = \frac{1}{5}x^5 - x^4 - \frac{5}{3}x^3 + 17$.

(a) Find the critical numbers of $f$.

$f'(x)$ IS DEFINED EVERYWHERE. (IT'S A POLYNOMIAL!)

$f'(x) = x^4 - 4x^3 - 5x^2 = x^2(x-5)(x+1)$

$f'(x) = 0 \Rightarrow \boxed{x = 0,\ x = 5,\ x = -1}$

(b) Use calculus techniques to find open intervals on which $f$ is increasing/decreasing.

| | | | | |
|---|---|---|---|---|
| SIGNS OF $f'(x)$ | $+$ | $-$ | $-$ | $+$ |
| | $-1$ | $0$ | $5$ | |
| $f$ IS... | INC | DEC | DEC | INC |

$f$ IS DECREASING ON $(-1,0) \cup (0,5)$

$f$ IS INCREASING ON $(-\infty,-1) \cup (5,\infty)$.

(c) Identify all relative extreme values.

FROM THE NUMBER LINE...

$f(-1) = \frac{262}{15}$ IS A REL. MAX.

$f(5) = -\frac{574}{3}$ IS A REL. MIN.

$f(0) = 17$ IS NEITHER A MIN NOR A MAX.

15. (5 points) Use the 2nd derivative to determine whether the graph of $r(x) = x^3 + \sin(10x)$ is concave up or concave down at the point where $x = 0.65$.

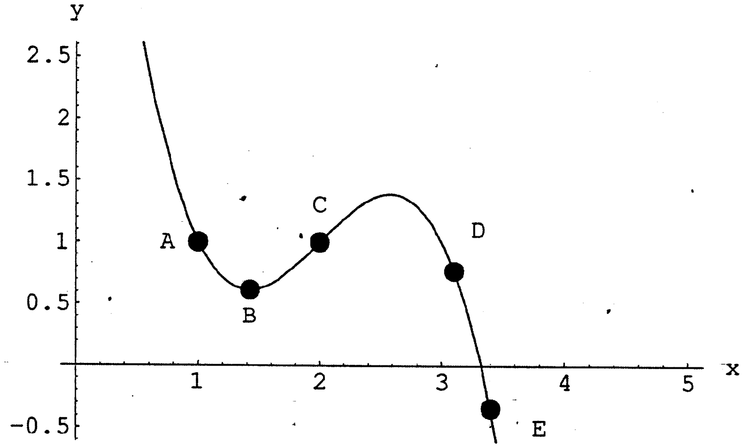
$r'(x) = 3x^2 + 10\cos(10x)$

$r''(x) = 6x - 100\sin(10x)$

$r''(0.65) = 6(0.65) - 100\sin(6.5) \approx -17.612$

GRAPH IS CONCAVE DOWN.

16. (6 points) The graph of $f$ is shown below. For each part of this problem, find a labeled point that satisfies the given condition.




(a) $f''(x) = 0$

C (Point of Inflection)

(b) $f'(x) = 0$

B (Horizontal Tangent Line)

(c) $f''(x) < 0$

D, E (Concave Down)

(d) $f(x) < 0$

E (Graph Below X-Axis)

(e) $f'(x) > 0$

C (Function Increasing)

(f) $f''(x) > 0$

A, B (Concave Up)