Math 131 - Final Exam

May 8, 2024

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Use algebraic techniques (not a graph, table, or L'Hôpital's rule) to determine each limit. You may need to use $+\infty$, $-\infty$, or DNE.

   (a) $\lim\limits_{y\to 0} \dfrac{y}{(y+6)^2-36}$

   0/0

   $$= \lim_{y\to 0} \frac{y}{y^2+12y} = \lim_{y\to 0} \frac{1}{y+12} = \boxed{\frac{1}{12}}$$

   (b) $\lim\limits_{x\to 3} \dfrac{x^2+9}{(x^2-9)^2}$

   18/0 Some kind of inf limit

   Both numerator and denom are positive around $x = 3$

   $\Rightarrow$ $\boxed{\text{Limit is } +\infty}$

2. (10 points) Carefully explain why f is NOT continuous at $x = 4$. Then state the type of discontinuity and further explain how f could be made continuous at that point.

$$f(x) = \begin{cases} x^2-5x+5, & x < 4 \\ 4\cos(\pi x)/x, & x > 4 \end{cases}$$

f is discontinuous at $x = 4$ because it is not defined at $x = 4$.

Since

$$\lim_{x\to 4^-} f(x) = 16-20+5 = 1$$

$$= \lim_{x\to 4^+} f(x) = \frac{4}{4} = 1,$$

the discont is removable.

To make f continuous at $x = 4$, simply define $f(4) = 1$.

3. (10 points) Let $f(x) = 3x - x^2$. Use the **limit definition of the derivative** to determine $f'(x)$. Show all work.

$$f'(x) = \lim_{h\to 0} \frac{\left[3(x+h) - (x+h)^2\right] - \left[3x - x^2\right]}{h}$$

$$= \lim_{h\to 0} \frac{3x + 3h - x^2 - 2xh - h^2 - 3x + x^2}{h}$$

$$= \lim_{h\to 0} \frac{3h - 2xh - h^2}{h} = \lim_{h\to 0} (3 - 2x - h) = \boxed{3 - 2x}$$

4. (10 points) Use basic differentiation rules to determine each derivative. Do not simplify.

(a) $\dfrac{d}{dx}\left(\dfrac{e^{2x}}{x^2+1}\right) = \dfrac{(x^2+1)(2e^{2x}) - (e^{2x})(2x)}{(x^2+1)^2}$

(b) $\dfrac{d}{dw} \sin^{-1}(\sqrt{w}) = \dfrac{1}{\sqrt{1 - (\sqrt{w})^2}} \cdot \dfrac{d}{dw}(\sqrt{w})$

$$= \frac{1}{\sqrt{1-w}} \cdot \frac{1}{2\sqrt{w}}$$

5. (10 points) Use logarithmic differentiation to compute dy/dx.

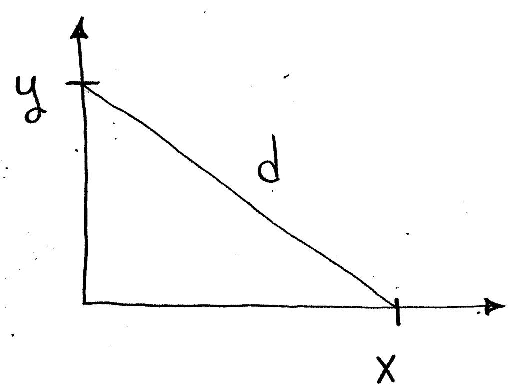
$$y = \frac{x^2+1}{x^2(x+3)}$$

$$\ln y = \ln(x^2+1) - 2\ln x - \ln(x+3)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1} - \frac{2}{x} - \frac{1}{x+3}$$

$$\frac{dy}{dx} = \left(\frac{x^2+1}{x^2(x+3)}\right)\left(\frac{2x}{x^2+1} - \frac{2}{x} - \frac{1}{x+3}\right)$$

6. (10 points) Two cars are driving away from an intersection, one heading due east and the other heading due north. When both cars are 2 miles from the intersection, the eastbound car is traveling at 50 mph and the distance between the cars is increasing at 80 mph. Find the speed of the northbound car.



$$\frac{dx}{dt} = 50$$

$$\frac{dd}{dt} = 80$$

FIND $\frac{dy}{dt}$ WHEN

$x = y = 2.$

$$2^2 + 2^2 = d^2 \Rightarrow d^2 = 8 \Rightarrow d = \sqrt{8}$$

$$x^2 + y^2 = d^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2d\frac{dd}{dt}$$

$$2(2)(50) + 2(2)\frac{dy}{dt} = 2\sqrt{8}(80)$$

$$\frac{dy}{dt} = \frac{2\sqrt{8}(80) - 2(2)(50)}{(2)(2)}$$

$$= 80\sqrt{2} - 50$$

$$\approx 63.137 \text{ mph}$$

7. (10 points) Let $f(x) = x^4 - 4x^3$.

(a) Use the 1st derivative test to determine all relative extreme values.

$f'(x) = 4x^3 - 12x^2 = 0$

$\Downarrow$

$4x^2(x-3) = 0$

$\underbrace{x = 0, \; x = 3}_{\text{Crit. \#'s.}}$

Signs of f': — (0) — (3) +

f is: DEC (0) DEC (3) INC

$f(3) = -27$ IS A REL MIN

$f(0) = 0$ IS NEITHER REL MIN NOR MAX

(b) Use the 2nd derivative test to determine open intervals on which the graph of f is concave up/down.

$f''(x) = 12x^2 - 24x = 0$

$\Downarrow$

$12x(x-2) = 0$

$\underbrace{x = 0, \; x = 2}_{\text{PIPs.}}$

Signs of f'': + (0) — (2) +

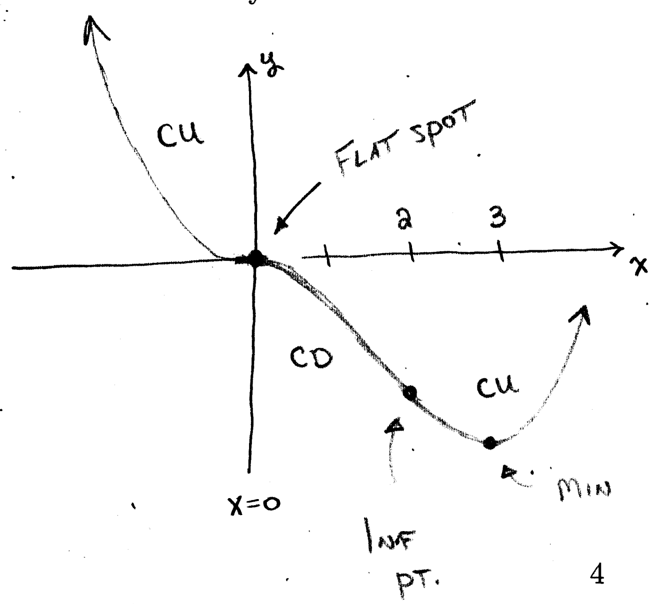
Graph is: CU (0) CD (2) CU.

GRAPH IS CONCAVE UP ON $(-\infty, 0) \cup (2, \infty)$.

GRAPH IS CONCAVE DOWN ON $(0, 2)$.

$f(2) = -16$, $f(0) = 0$

(c) Draw a rough sketch of the graph of f being sure that your graph shows the features you've found above.



8. (10 points) Use any analytical method (not a table or graph) to determine each limit.

(a) $\lim_{x\to\infty} 5xe^{-x}$ $\infty \cdot 0$

$5/\infty$

$$\lim_{x\to\infty} \frac{5x}{e^x} = \lim_{x\to\infty} \frac{5}{e^x} = \boxed{0}$$

(b) $\lim_{x\to 0} \left(\frac{x^2 - 2 + 2\cos x}{x^4}\right)$ $0/0$

$0/0$ $0/0$

$$= \lim_{x\to 0} \frac{2x - 2\sin x}{4x^3} = \lim_{x\to 0} \frac{2 - 2\cos x}{12x^2}$$

$$= \lim_{x\to 0} \frac{2\sin x}{24x} = \frac{2}{24} = \frac{1}{12}$$

9. (10 points) For each part of this problem, set up and evaluate an appropriate definite integral.

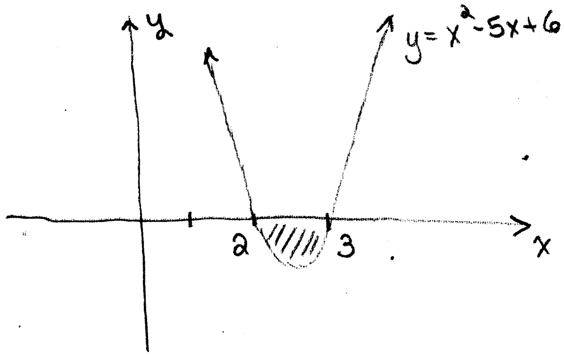
(a) Find the average value of $f(x) = \frac{2}{x} + \sqrt[4]{x}$ on $[1, 16]$.

$$\frac{1}{15}\int_1^{16}\left(\frac{2}{x} + x^{1/4}\right)dx = \frac{1}{15}\left[2\ln|x| + \frac{4}{5}x^{5/4}\right]_1^{16}$$

$$= \frac{1}{15}\left[2\ln 16 + \frac{4}{5}(32)\right] - \frac{1}{15}\left[2(0) + \frac{4}{5}\right]$$

$$= \frac{1}{15}\left[2\ln 16 + \frac{124}{5}\right] \approx 2.023$$

(b) Find the area of the fourth quadrant region between the x-axis and the graph of $y = \underbrace{x^2 - 5x + 6}_{(x-2)(x-3)}$.



$$\text{AREA} = -\int_2^3 (x^2 - 5x + 6)\,dx$$

$$= -\left(\frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x\right)\Big|_2^3$$

$$= -\left[\left(9 - \frac{45}{2} + 18\right) - \left(\frac{8}{3} - 10 + 12\right)\right]$$

$$= \boxed{\frac{1}{6}}$$

10. (10 points) Evaluate each integral. (You may need to use a substitution.)

(a) $\int \left(e^x + \frac{1}{1+x^2} + \sec^2 x\right) dx$

$$= \boxed{e^x + \tan^{-1} x + \tan x + C}$$

(b) $\int_0^\pi \cos^4 x \sin x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$$= -\int_1^{-1} u^4 \, du = \int_{-1}^{1} u^4 \, du$$

$$= \frac{1}{5}u^5 \Big|_{-1}^{1} = \boxed{\frac{2}{5}}$$