Math 131 - Test 3

April 15, 2025

Name Key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Let $f(x) = x^2 - 2x - 3$.

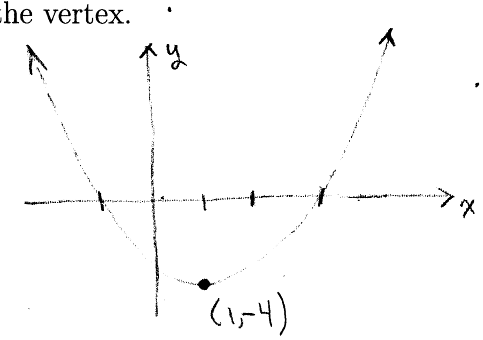
   (a) Sketch a rough graph of f. Even though your graph may be rough, clearly indicate the coordinates of the vertex.

$f(x) = (x-3)(x+1)$

Vertex at $x = \frac{3-1}{2}$

$x = 1$

$f(1) = -4$



   (b) Explain why f does not have an inverse.

f is not one-to-one.

Its graph fails the horizontal line test.

   (c) With the restriction to $x \geq 1$, f now has an inverse. With that restriction in mind, compute $f^{-1}(-3)$.

$f^{-1}(-3) = x \Leftrightarrow f(x) = -3$

$x^2 - 2x - 3 = -3$

$x - 2x = 0$

$x(x-2) = 0 \Rightarrow \cancel{x=0}, \; x = 2$

$\boxed{f^{-1}(-3) = 2}$

   (d) Find $(f^{-1})'(-3)$.

$$\frac{1}{f'(f^{-1}(-3))} = \frac{1}{f'(2)} = \boxed{\frac{1}{2}}$$

$f'(x) = 2x - 2$

$f'(2) = 4 - 2 = 2$

2. (6 points) Suppose g and g^{-1} are differentiable functions. The table below shows the values of $g(x)$ and $g'(x)$ at selected values of x. Find $(g^{-1})'(5)$. Show how you got it.

x	2	3	4	5
$g(x)$	3	5	6	9
$g'(x)$	2	1	3	7

$$(g^{-1})'(5) = \frac{1}{g'(g^{-1}(5))}$$

$g^{-1}(5) = 3$

$g'(3) = 1$

$$= \frac{1}{g'(3)} = \frac{1}{1} = \boxed{1}$$

3. (6 points) Let $g(x) = 4x^2 \sin^{-1} x$. Find $g'(x)$. Then find the exact value of $g'(1/2)$. (Do not give a decimal number for your answer.)

$$\boxed{g'(x) = 8x \sin^{-1} x + \frac{4x^2}{\sqrt{1-x^2}}}$$

$$g'\left(\tfrac{1}{2}\right) = 4 \sin^{-1}\left(\tfrac{1}{2}\right) + \frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{4\pi}{6} + \frac{1}{\sqrt{3/4}}$$

$$= \boxed{\frac{2\pi}{3} + \frac{2}{\sqrt{3}}}$$

4. (6 points) Find an equation of the line tangent to the graph of $y = \ln(x^2 e^{x+3})$ at the point where $x = 1$.

$y = \ln x^2 + \ln e^{x+3}$

$y = 2 \ln x + x + 3$

$$\frac{dy}{dx} = \frac{2}{x} + 1$$

Slope: $\left.\frac{dy}{dx}\right|_{x=1} = 3$

Point: $x = 1$

$y = \ln(e^4) = 4$

Tangent Line:

$y - 4 = 3(x-1)$

or

$y = 3x + 1$

5. (6 points) Find the instantaneous rate of change of $h(x) = \dfrac{2}{e^{2x} + e^{-2x}}$ at the point where $x = 0$.

$$h'(x) = \frac{0(e^{2x}+e^{-2x}) - 2(2e^{2x} - 2e^{-2x})}{(e^{2x}+e^{-2x})^2} = \frac{-4(e^{2x} - e^{-2x})}{(e^{2x}+e^{-2x})^2}$$

$$h'(0) = \frac{-4(1-1)}{(1+1)^2} = \boxed{0}$$

6. (2 points) Use the change-of-base formula to rewrite $\log_7 6$ in terms of natural logarithms.

$$\log_7 6 = \boxed{\frac{\ln 6}{\ln 7}}$$

7. (5 points) Determine the derivative of $g(x) = 5^{\tan x}$.

$$\boxed{g'(x) = 5^{\text{TAN } x}(\ln 5)(\sec^2 x)}$$

8. (8 points) Use logarithmic differentiation to find dy/dx when $y = \dfrac{\sqrt{x^2+1}}{x^5(8x+7)^2}$.

$$\ln y = \frac{1}{2}\ln(x^2+1) - 5\ln x - 2\ln(8x+7)$$

$$y \cdot \frac{1}{y}\frac{dy}{dx} = \left(\frac{1}{2}\frac{2x}{x^2+1} - \frac{5}{x} - 2\frac{8}{8x+7}\right)\cdot y$$

$$\boxed{\frac{dy}{dx} = \left[\frac{x}{x^2+1} - \frac{5}{x} - \frac{16}{8x+7}\right]\frac{\sqrt{x^2+1}}{x^5(8x+7)^2}}$$

9. (8 points) Determine the linearization of $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$ at $x = 4$. Then use your linearization to approximate $f(4.05)$.

$$f(4) = 2 + \frac{1}{2} = \frac{5}{2}$$

$$\boxed{L(x) = \frac{5}{2} + \frac{3}{16}(x-4)}$$

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}$$

$$f(4.05) \approx L(4.05) = \frac{5}{2} + \frac{0.15}{16}$$

$$f'(4) = \frac{1}{4} - \frac{1}{16}$$

$$\approx \boxed{2.509}$$

$$= \frac{3}{16}$$

10. (6 points) Use differentials to approximate the change in $f(x) = \tan^{-1} x$ as x changes from $x = 1$ to $x = 0.96$.

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$x = 1, \quad \Delta x = -0.04$$

$$\Delta y \approx \frac{1}{2}(-0.04) = -0.02$$

$$dy = \frac{1}{1+x^2}\,dx$$

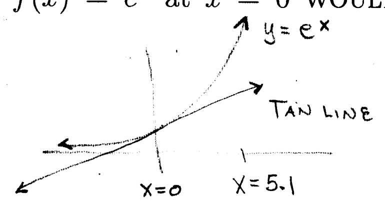
$$\boxed{\Delta y \approx -0.02}$$

$$\Delta y \approx \frac{1}{1+x^2}\,\Delta x$$

11. (6 points) Recall that a linearization is essentially a tangent line approximation. It might be useful to think graphically for the following problems.

(a) Briefly explain why the linearization of $f(x) = e^x$ at $x = 0$ WOULD NOT BE USEFUL for approximating $f(5.1)$.

LOOK AT THE GRAPHS OF $y = e^x$ AND ITS LINEARIZATION AT $x = 0$



e^x grows too quickly as x grows.

(b) Briefly explain why the linearization of $f(x) = 2x+1$ at $x = 0$ WOULD BE USEFUL for approximating $f(5.1)$.

$f(x)$ IS A LINEAR FUNCTION!

ITS LINEARIZATION IS ITSELF!

$\therefore f(x) = L(x)$ THEY WILL BE EQUAL EVERYWHERE.

12. (6 points) The length of the base of a right triangle is x cm. Its height is 5 cm more than its base, so that its area is given by

$$A(x) = \frac{1}{2}x(x+5).$$

Use differentials to approximate the change in the triangle's area when $x = 4$ cm and $\Delta x = 0.2$ cm. Include units with your answer.

$A(x) = \frac{1}{2}x^2 + \frac{5}{2}x$

$A'(x) = x + \frac{5}{2}$

$dA = \left(x + \frac{5}{2}\right) dx$

$\Delta A \approx \left(x + \frac{5}{2}\right) \Delta x$

$x = 4\text{ cm}, \quad \Delta x = 0.2\text{ cm}$

$\Delta A \approx \left(4 + \frac{5}{2}\right)(0.2)$

$= \boxed{1.3\text{ cm}^2}$

13. (4 points) Let $f(x) = \ln x$. Find $f'(x)$ and then explain why $x = 0$ IS NOT a critical number of f.

$f'(x) = \frac{1}{x}$

$f'(0)$ DNE, so $x=0$ may be a crit #.

Let's look at the domain of f.

Domain of f is $(0, \infty)$.

$x = 0$ is not in domain of f.

14. (8 points) Find the critical numbers of $\underbrace{f(x) = \frac{1}{4}x^4 - x^3 + x^2 + 1}_{\text{Defined for all real \#'s.}}$.

$f'(x) = x^3 - 3x^2 + 2x = x(x^2 - 3x + 2) = x(x-2)(x-1)$

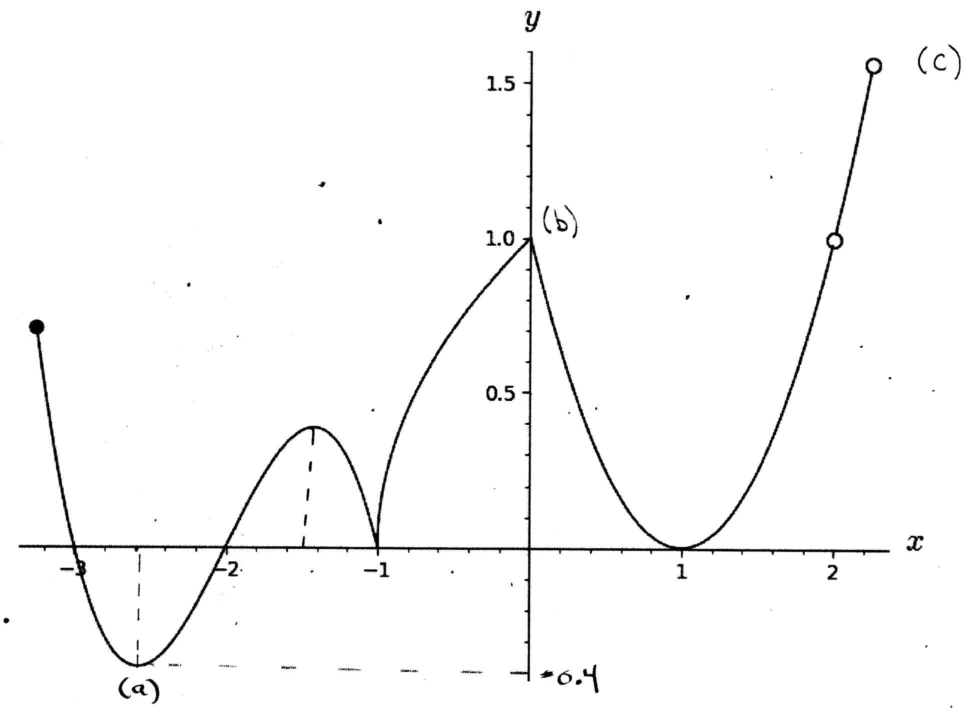
$f'(x)$ DNE nowhere

$f'(x) = 0 \Rightarrow x(x-2)(x-1) = 0$

$\Rightarrow \boxed{x=0, \; x=2, \; x=1}$

All are crit. #'s.

15. (13 points) The graph of g is shown below. Use the graph to solve the problems below. Notice that the domain of g is $[-3.25, 2) \cup (2, 2.25)$.



(a) Estimate the absolute minimum value of g and say where it occurs.

ABS MIN = -0.4 AT $X = -2.6$

(b) Estimate a relative maximum value of g and say where it occurs.

A REL MAX. IS $y = 1.0$ AT $X = 0$

(c) Explain why g has no absolute maximum value.

AS $X \to 2^-$ THE FUNCTION VALUES INCREASE AND APPROACH $y = 1.5$ BUT THEY DO NOT REACH $y = 1.5$.

(d) Estimate five critical numbers.

$X = -2.6$, $X = -1.5$, $X = -1$, $X = 0$, $X = 1$

FLAT SPOTS AND SHARP PTS.

(e) Explain why a relative maximum does not occur at the graph's left endpoint.

DOMAIN ENDPTS OR BOUNDARY PTS DO NOT COUNT AS LOCATION FOR RELATIVE EXTREMA. FUNCTION MUST BE DEFINED ON AN OPEN INTERVAL AROUND THE PT.

(f) Estimate a relative minimum value of g and say where it occurs.

A REL MIN IS $y = 0$ AT $X = 1$