Math 132 - Test 1

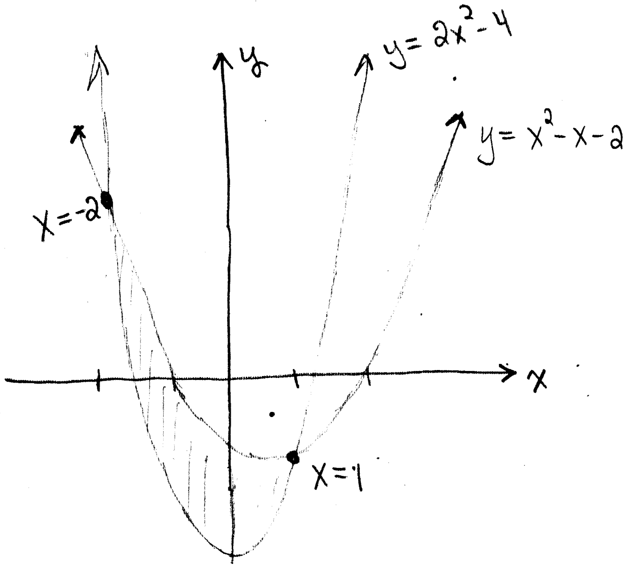
September 21, 2022

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary. Unless otherwise indicated, do all integration by hand.

1. (12 points) Find the area of the bounded region between the graphs of $y = 2x^2 - 4$ and $y = x^2 - x - 2 = (x-2)(x+1)$



$$2x^2 - 4 = x^2 - x - 2$$
$$\Downarrow$$
$$x^2 + x - 2 = (x+2)(x-1) = 0$$
$$\Downarrow$$
$$x = -2,\ x = 1$$

$$\text{Area} = \int_{-2}^{1} \left[(x^2 - x - 2) - (2x^2 - 4) \right] dx$$

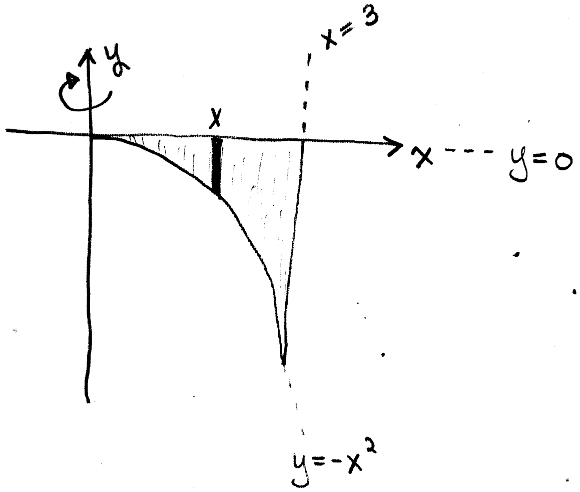
$$= \int_{-2}^{1} (-x^2 - x + 2)\, dx$$

$$= -\frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x \Big|_{-2}^{1}$$

$$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right)$$

$$= \boxed{\frac{9}{2}}$$

2. (10 points) The 4th-quadrant region bounded by the graphs of $y = -x^2$, $y = 0$, and $x = 3$ is rotated about the y-axis to form a solid. Find the volume of the solid.



SHELLS...

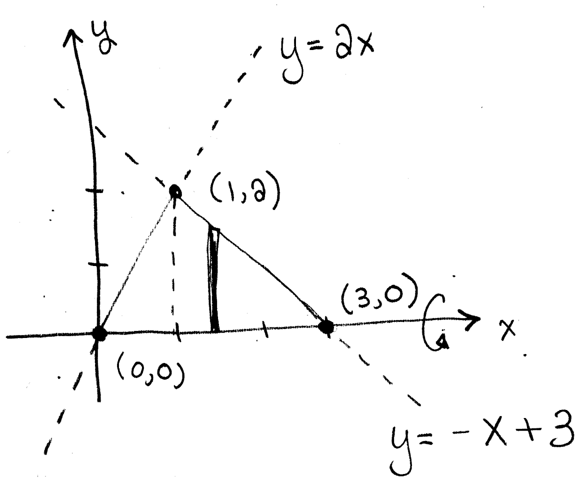
$$\text{Volume} = 2\pi \int_0^3 x\left[0 - (-x^2)\right] dx$$

$$= 2\pi \int_0^3 x^3\, dx = \frac{\pi}{2} x^4 \Big|_0^3$$

$$= \boxed{\frac{81\pi}{2}}$$

≈ 127.23

3. (12 points) A triangular region in the xy-plane has vertices at $(0, 0)$, $(1, 2)$, and $(3, 0)$. The region is rotated about the x-axis to form a solid. Use our calculus techniques to find the volume of the solid.



DISKS...

$$\text{Volume} = \pi \int_0^1 (2x)^2\, dx + \pi \int_1^3 (-x+3)^2\, dx$$

$$= \pi \int_0^1 4x^2\, dx + \pi \int_1^3 (x^2 - 6x + 9)\, dx$$

$$= \pi \left(\frac{4}{3} x^3 \Big|_0^1\right) + \pi \left(\frac{1}{3}x^3 - 3x^2 + 9x\right)\Big|_1^3$$

$$= \pi \cdot \left(\frac{4}{3}\right) + \pi \left[(9 - 27 + 27) - \left(\frac{1}{3} - 3 + 9\right)\right]$$

$$= \boxed{4\pi}$$

4. (10 points) Set up the definite integral that gives the length of the graph of $y = x^{3/2}$ from $x = 0$ to $x = 1$. Evaluate your definite integral by hand, showing all work.

$$y = x^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2}x^{1/2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{9}{4}x$$

$$\text{ARC LENGTH} = \int_0^1 \sqrt{1 + \frac{9}{4}x}\, dx = \frac{4}{9}\int_1^{13/4} u^{1/2}\, du$$

$u = 1 + \frac{9}{4}x$

$du = \frac{9}{4}dx$

$\frac{4}{9}du = dx$

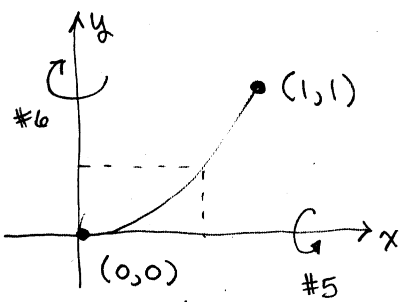
$x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = \frac{13}{4}$

$$= \frac{8}{27}u^{3/2}\Big|_1^{13/4}$$

$$= \boxed{\frac{8}{27}\left[\left(\frac{13}{4}\right)^{3/2} - 1\right] \approx 1.44}$$

5. (5 points) The graph of $y = x^{3/2}$ from $x = 0$ to $x = 1$ is rotated about the x-axis to form a surface. Set up the definite integral that gives the area of the surface. Then use your calculator to approximate the value of the integral.



$$\text{AREA} = 2\pi\int_0^1 x^{3/2}\sqrt{1 + \frac{9}{4}x}\, dx$$

$$\approx 1.2857\pi$$

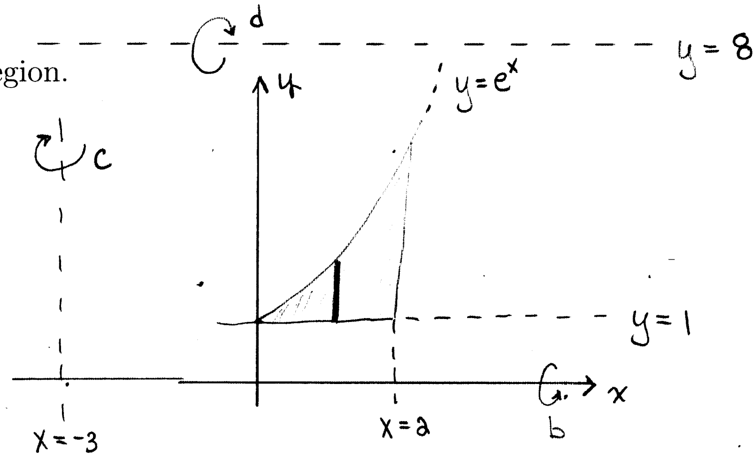
$$\approx 4.0391$$

6. (3 points) The graph of $y = x^{3/2}$ from $x = 0$ to $x = 1$ is rotated about the y-axis to form a surface. Set up the definite integral that gives the area of the surface. Do not evaluate the integral.

$$2\pi\int_0^1 x\sqrt{1 + \frac{9}{4}x}\, dx$$

7. (16 points) Consider the 1st-quadrant region bounded by the graphs of $y = e^x$, $y = 1$, and $x = 2$.

(a) Sketch the region.



(b) The region is rotated about the x-axis to form a solid. Set up, **but do not evaluate**, a definite integral that gives the volume of the solid.

WASHERS ...

$$\text{Volume} = \pi \int_0^2 \left[(e^x)^2 - (1)^2\right] dx = \boxed{\pi \int_0^2 (e^{2x} - 1)\, dx}$$

(c) The region is rotated about the line $x = -3$ to form a solid. Set up, **but do not evaluate**, a definite integral that gives the volume of the solid.

SHELLS ...

$$\text{Volume} = \boxed{2\pi \int_0^2 (x+3)(e^x - 1)\, dx}$$

(d) The region is rotated about the line $y = 8$ to form a solid. Set up, **but do not evaluate**, a definite integral that gives the volume of the solid.

WASHERS ...

$$\text{Volume} = \pi \int_0^2 \left[(8-1)^2 - (8-e^x)^2\right] dx$$

$$= \boxed{\pi \int_0^2 \left[49 - (8-e^x)^2\right] dx}$$

ANOTHER APPROACH: $\int_0^{17} 4.25(17-y)\,dy + 17 \times 75$

8. (10 points) Chain that weighs 4.25 lb/ft is used to lift a 75-lb rock out of a hole that is 17 ft deep. Determine the work required.

17 FT OUT: $y = 0,\ W = 147.25$

0 FT OUT: $y = 17,\ W = 75$

SLOPE $= -4.25$

$W = -4.25y + 147.25$

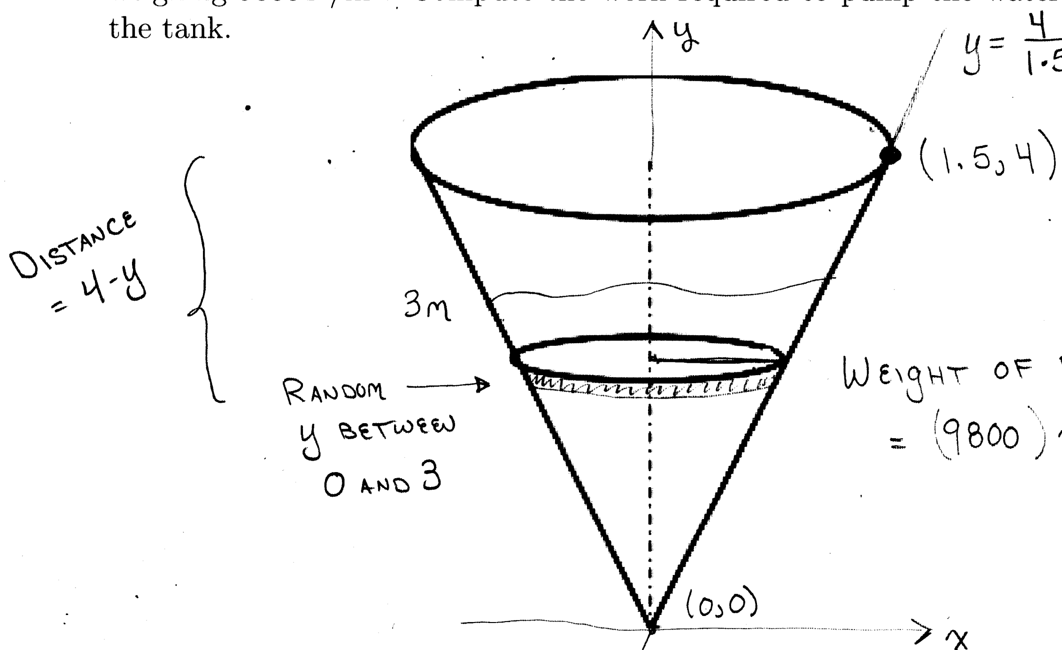
$$\text{WORK} = \int_0^{17} (-4.25y + 147.25)\,dy$$

$$= -\frac{4.25}{2}y^2 + 147.25y \Big|_0^{17}$$

$$= \boxed{1889.125 \text{ FT LB}}$$

9. (10 points) A tank has the shape of a circular cone with its point down. The tank is 4 m tall, and its radius at the top is 1.5 m. It is filled to a height of 3 m with water weighing 9800 N/m^3. Compute the work required to pump the water over the top of the tank.

$y = \frac{4}{1.5}x \Rightarrow x = \frac{1.5}{4}y$



WEIGHT OF DISK AT y

$= (9800)\pi\left(\frac{1.5}{4}y\right)^2 dy$

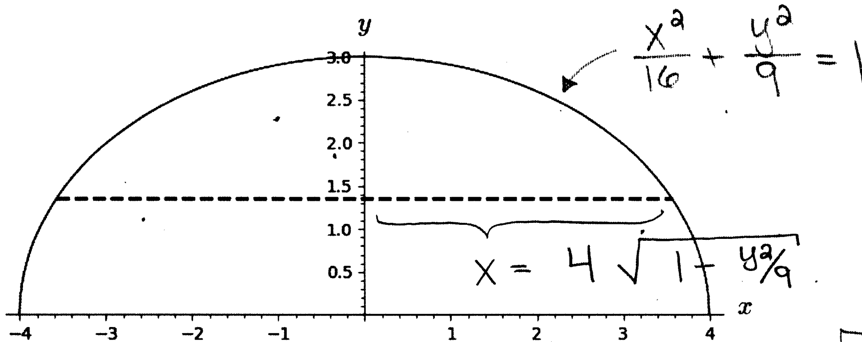
$$\text{WORK} = \int_0^3 9800\pi\left(\frac{1.5}{4}y\right)^2(4-y)\,dy = 9800\pi\left(\frac{2.25}{16}\right)\int_0^3 (4y^2 - y^3)\,dy$$

$$1378.125\pi\left(\frac{4}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^3 = 1378.125\pi\left(36 - \frac{81}{4}\right)$$

$$= 21705.46875\pi$$

$$\approx \boxed{68{,}189.74 \text{ N-m}}$$

10. (12 points) The base of a solid lies in the xy-plane over the region bounded by the upper half of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ (see the graph below). Each cross section of the solid perpendicular to the y-axis is a rectangle whose height at y is y. (A dashed strip is shown at a random y-value.) Find the volume of the solid.



$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$x = 4\sqrt{1 - y^2/9}$$

$$\text{BASE} = 2 \cdot 4\sqrt{1 - y^2/9}$$

AREA OF RECTANGLE AT

$$y = \text{BASE} \times \text{HEIGHT}$$

$$= 8 \cdot \sqrt{1 - y^2/9} \cdot y$$

$$\text{VOLUME} = \int_0^3 8\sqrt{1 - \frac{y^2}{9}}\, y\, dy$$

$$u = 1 - \frac{y^2}{9} \qquad y = 0 \Rightarrow u = 1$$

$$du = -\frac{2}{9} y\, dy \qquad y = 3 \Rightarrow u = 0$$

$$-\frac{9}{2}\, du = y\, dy$$

VOLUME =

$$\frac{9}{2}\int_0^1 8u^{1/2}\, du = 36 \cdot \frac{2}{3} u^{3/2}\Big|_0^1 = 36 \cdot \frac{2}{3} = \boxed{24 \text{ UNITS}^2}$$