Math 132 - Test 2
October 19, 2022

Name Key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) Use the definitions of the hyperbolic functions (in terms of exponential functions) to find the exact value of each of the following.

   (a) $\cosh(0)$

   $$= \frac{e^0 + e^{-0}}{2} = \frac{2}{2} = \boxed{1}$$

   (b) $\sinh(\ln 2)$

   $$= \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{3/2}{2} = \boxed{\frac{3}{4}}$$

   (c) $\tanh(1)$

   $$\frac{e^1 - e^{-1}}{e^1 + e^{-1}} = \frac{e^2 - 1}{e^2 + 1} \approx 0.7616$$

   (d) $\text{sech}(0)$

   $$= \frac{2}{e^0 + e^{-0}} = \frac{1}{\cosh(0)} = \boxed{1}$$

2. (4 points) Use the definitions of the hyperbolic functions (in terms of exponential functions) to show that

   $$\tanh^2(x) + \text{sech}^2(x) = 1.$$

$$\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 + \left(\frac{2}{e^x + e^{-x}}\right)^2 = \frac{e^{2x} - 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} + \frac{4}{e^{2x} + 2 + e^{-2x}}$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{e^{2x} + 2 + e^{-2x}} = 1 \checkmark$$

3. (4 points) Starting with the definition of $\coth(x)$ (in terms of exponential functions), show that $\coth(x) = \dfrac{e^{2x}+1}{e^{2x}-1}$. Then compute $\lim\limits_{x\to\infty} \coth(x)$.

$$\text{COTH}(x) = \frac{(e^{x}+e^{-x})}{(e^{x}-e^{-x})} \cdot \frac{e^{x}}{e^{x}} = \frac{e^{2x}+1}{e^{2x}-1} \quad \checkmark$$

$$\lim_{x\to\infty} \frac{e^{2x}+1}{e^{2x}-1} \overset{\infty/\infty}{=} \lim_{x\to\infty} \frac{2e^{2x}}{2e^{2x}} = \lim_{x\to\infty} 1 = \boxed{1}$$

L'Hôpital's Rule

4. (4 points) Evaluate the indefinite integral: $\displaystyle\int x^2 \cosh(x^3)\, dx$

$$u = x^3$$

$$du = 3x^2\, dx$$

$$\frac{1}{3}\, du = x^2\, dx$$

$$\frac{1}{3}\int \text{COSH}(u)\, du = \frac{1}{3}\text{SINH}(u) + C$$

$$= \boxed{\frac{1}{3}\text{SIN}(x^3) + C}$$

5. (8 points) Evaluate the indefinite integral: $\displaystyle\int 7x^2 e^{5x}\, dx$

SIGNS	U AND DERIVS	dv/dx AND ANTI-DS
+	$7x^2$	e^{5x}
—	$14x$	$\frac{1}{5}e^{5x}$
+	14	$\frac{1}{25}e^{5x}$
—	0	$\frac{1}{125}e^{5x}$

$$= \boxed{\left(\frac{7}{5}x^2 - \frac{14}{25}x + \frac{14}{125}\right)e^{5x} + C}$$

6. (8 points) Evaluate the indefinite integral: $\displaystyle\int \sec^6 x\,dx$

$$\int \sec^4 x \sec^2 x \, dx$$

$$= \int (1+\tan^2 x)^2 \sec^2 x \, dx = \int (1+u^2)^2 \, du$$

$u = \tan x$

$du = \sec^2 x \, dx$

$$= \int (1 + 2u^2 + u^4) \, du$$

$$= u + \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= \tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$$

7. (8 points) Evaluate the indefinite integral: $\displaystyle\int \frac{\sin^3 y}{\cos^4 y}\,dy$

$$\int \frac{\sin^2 y \sin y \, dy}{\cos^4 y} = \int \frac{(1-\cos^2 y)\sin y \, dy}{\cos^4 y}$$

$u = \cos y$

$du = -\sin y \, dy$

$-du = \sin y \, dy$

$$= -\int \frac{1-u^2}{u^4} \, du$$

$$= \int \frac{1}{u^2} - \frac{1}{u^4} \, du = \int (u^{-2} - u^{-4}) \, du$$

$$= \frac{-1}{u} + \frac{1}{3u^3} + C$$

$$= -\frac{1}{\cos y} + \frac{1}{3\cos^3 y} + C$$

8. (8 points) Assume k is some positive constant. Evaluate the following indefinite integral, which arises in certain population growth models.

$$\int \frac{dP}{kP - P^2}$$

$$\frac{1}{kP - P^2} = \frac{1}{P(k-P)} = \frac{A}{P} + \frac{B}{k-P}$$

$$1 = A(k-P) + BP$$

$$P = k \Rightarrow 1 = Bk \Rightarrow B = \frac{1}{k}$$

$$P = 0 \Rightarrow 1 = Ak \Rightarrow A = \frac{1}{k}$$

$$\int \left(\frac{1/k}{P} + \frac{1/k}{k-P} \right) dP = \boxed{\frac{1}{k} \ln|P| - \frac{1}{k} \ln|k-P| + C}$$

9. (6 points) In the following integral, carry out the appropriate trigonometric substitution, simplify the integrand, and then stop. Do not evaluate the new integral.

$$\int \frac{8\,dx}{(4x^2 + 1)^2}$$

$$2x = \tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

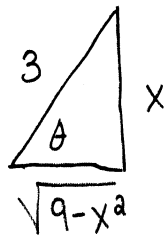
$$2\,dx = \sec^2\theta \, d\theta$$

$$\int \frac{4\sec^2\theta \, d\theta}{(\tan^2\theta + 1)^2} = \int \frac{4\sec^2\theta \, d\theta}{\sec^4\theta} = \int \frac{4 \, d\theta}{\sec^2\theta}$$

$$= \underline{\underline{\int 4\cos^2\theta \, d\theta}}$$

10. (6 points) After making the trigonometric substitution $x = 3\sin\theta$, you evaluated an integral and obtained $\frac{9}{2}(\theta - \sin\theta\cos\theta) + C$. Resubstitute and write your result in terms of the variable x.

$\frac{x}{3} = \sin\theta$



$$\frac{9}{2}(\theta - \sin\theta\cos\theta) + C = \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\,\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \boxed{\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9-x^2} + C}$$

11. (3 points) Write the form of the partial fraction decomposition of $\frac{x^3 - 7}{(x^2+4)^2(3x-1)(x+7)^3}$. Do not solve for the undetermined coefficients.

$$\boxed{\frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{E}{3x-1} + \frac{F}{x+7} + \frac{G}{(x+7)^2} + \frac{H}{(x+7)^3}}$$

12. (8 points) Evaluate the indefinite integral: $\displaystyle\int \frac{5x^2+3}{x^3+x}\,dx$

$$\frac{5x^2+3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$5x^2+3 = A(x^2+1) + (Bx+C)x$$

x^2: $A+B=5$

x: $C=0$

1: $A=3$

$B=2$

$$\int\left(\frac{3}{x} + \frac{2x}{x^2+1}\right)dx$$

$u = x^2+1$

$du = 2x\,dx$

$$\int \frac{2x}{x^2+1}\,dx = \int \frac{1}{u}\,du = \ln|u|$$

$$\int\left(\frac{3}{x} + \frac{2x}{x^2+1}\right)dx = \boxed{3\ln|x| + \ln|x^2+1| + C}$$

Take-home portion: The following problems are due no later than October 24 at 2:30 pm. You must work individually on these problems and show all work. No credit will be given for group work or a tutor's work.

13. (9 points) Evaluate the following definite integral. Write your answer in exact form.

$$\int_0^1 \ln(x^2+1)\,dx$$

$u = \ln(x^2+1)$ $\quad du = \frac{2x}{x^2+1}\,dx.$

$dv = dx$ $\quad v = x$

ADD & SUBTRACT 2. ESSENTIALLY LONG DIVISION.

$$x\ln(x^2+1)\Big|_0^1 - \int_0^1 \frac{2x^2}{x^2+1}\,dx = \ln 2 - \underbrace{\int_0^1 \frac{2x^2+2}{x^2+1}\,dx + \int \frac{2}{x^2+1}\,dx}$$

$$= \ln 2 - \int_0^1 2\,dx + 2\tan^{-1}x\Big|_0^1$$

$$= \ln 2 - 2 + 2\tan^{-1}(1) - 2\tan^{-1}(0)$$

$$= \ln 2 - 2 + \frac{\pi}{2} - 0$$

$$= \boxed{\frac{\pi}{2} - 2 + \ln 2}$$

14. (7 points) Evaluate the indefinite integral: $\displaystyle\int 16 \sin^2 x \cos^2 x \, dx$

$$16 \sin^2 x \cos^2 x = 16\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)$$

$$= 4(1 - \cos 2x)(1 + \cos 2x) = 4(1 - \cos^2 2x) = 4 \sin^2 2x$$

$$= 2 - 2\cos 4x$$

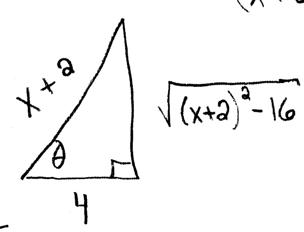
$$\int (2 - 2\cos 4x)\, dx = \boxed{2x - \frac{1}{2}\sin 4x + C}$$

15. (9 points) Assume $x > 2$ and evaluate $\displaystyle\int \frac{1}{\underbrace{\sqrt{x^2 + 4x - 12}}_{(x+2)^2 - 16}}\, dx.$

(Helpful advice: Complete the square.)

$x + 2 = 4 \sec\theta$

$dx = 4 \sec\theta \tan\theta$



$x > 2 \Rightarrow \sec\theta > 1$

$\Rightarrow 0 \le \theta < \frac{\pi}{2}$

$$\int \frac{4\sec\theta \tan\theta \, d\theta}{\sqrt{16\sec^2\theta - 16}} = \int \frac{4\sec\theta\tan\theta\, d\theta}{\sqrt{16\tan^2\theta}} = \int \frac{4\sec\theta\tan\theta}{4|\tan\theta|}\, d\theta$$

$|\tan\theta| = \tan\theta$

For $0 \le \theta < \frac{\pi}{2}$

$$= \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \boxed{\ln\left|\frac{x+2}{4} + \frac{\sqrt{(x+2)^2 - 16}}{4}\right| + C}$$