Math 132 - Test 3
November 16, 2022

Name Key

Score ________

Show all work to receive full credit. Supply explanations where necessary. You must work individually on this test. Please do not confuse **sequences** and **series**. The test is due November 28, 2022 at 12:30 pm.

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1. (5 points) Use the data given below and the trapezoid rule with $n = 8$ to approximate $\int_0^2 f(x)\,dx$.

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
$f(x)$	4.27	4.38	4.61	5.84	6.12	7.24	7.60	8.15	8.28

$$h = \frac{2-0}{8} = 0.25$$

$$\int_0^2 f(x)\,dx \approx \frac{0.25}{2}\Big[4.27 + 2(4.38) + 2(4.61) + 2(5.84) + 2(6.12) + 2(7.24) + 2(7.60) + 2(8.15) + 8.28\Big]$$

$$= \boxed{12.55375}$$

2. (5 points) Use Simpson's rule with $n = 6$ to approximate $\int_1^3 \sin\sqrt{x}\,dx$.

$$h = \frac{3-1}{6} = \frac{1}{3}; \quad x_0 = 1,\ x_1 = \frac{4}{3},\ x_2 = \frac{5}{3},\ \ldots,\ x_6 = \frac{9}{3}$$

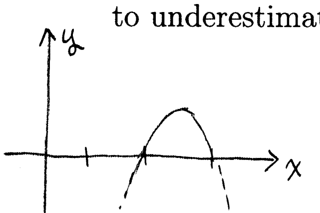
$$\int_1^3 \sin\sqrt{x}\,dx \approx \frac{1/3}{3}\Big[\sin\sqrt{1} + 4\sin\sqrt{\tfrac{4}{3}} + 2\sin\sqrt{\tfrac{5}{3}} + 4\sin\sqrt{\tfrac{6}{3}} + 2\sin\sqrt{\tfrac{7}{3}} + 4\sin\sqrt{\tfrac{8}{3}} + \sin\sqrt{3}\Big]$$

$$\approx \frac{1}{9}(17.3508711) \approx \boxed{1.92787}$$

$$-x^2 + 5x - 6 = -(x-2)(x-3)$$

3. (5 points) Consider the definite integral $\int_2^3 (-x^2 + 5x - 6)\, dx$.

(a) Thinking about the graph of the integrand, would you expect the trapezoid rule to underestimate or overestimate the value of the integral. Explain.



THE GRAPH IS CONCAVE DOWN.

THE TRAPEZOIDS WILL ALL BE UNDER THE CURVE.

THE RULE WILL GIVE AN $\boxed{\text{UNDERESTIMATE.}}$

(b) What about Simpson's rule?

THE INTEGRAND IS A DEGREE-2 POLYNOMIAL. SIMPSON'S RULE WILL BE EXACTLY CORRECT.

4. (15 points) Consider the integral $\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$.

(a) Explain why/how this integral is improper.

THE INTEGRATION INTERVAL IS INFINITE

-AND- THE INTEGRAND HAS AN INFINITE DISCONTINUITY AT $x = 0$.

(b) Evaluate the indefinite integral $\int \frac{dx}{\sqrt{x}(x+1)}$. (Hint: Use $u = \sqrt{x}$.)

$u = x^{1/2}$

$du = \frac{1}{2}x^{-1/2}dx$

$2\,du = x^{-1/2}dx$

$x = u^2$

$$\int \frac{2\,du}{u^2+1} = 2\tan^{-1} u + C = \boxed{2\tan^{-1}\sqrt{x} + C}$$

(c) Write the original improper integral with limits and evaluate.

$$\lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}(x+1)} + \lim_{b\to\infty} \int_1^b \frac{dx}{\sqrt{x}(x+1)}$$

$$= \lim_{a\to 0^+} \left(2\tan^{-1}\sqrt{x}\right)\Big|_a^1 + \lim_{b\to\infty} \left(2\tan^{-1}\sqrt{x}\right)\Big|_1^b$$

$$2\tan^{-1}(1) - 2\tan^{-1}(0) + 2\lim_{b\to\infty} \tan^{-1}\sqrt{x} - 2\tan^{-1}(1)$$

$$= 2\left(\frac{\pi}{4}\right) - 2(0) + 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{4}\right) = \boxed{\pi}$$

5. (5 points) Consider the sequence whose nth term is $a_n = n \sin\left(\frac{1}{n}\right)$. Determine whether the sequence converges or diverges. If it converges, find its limit.

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) \overset{\infty\cdot 0}{=} \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{u\to 0^+} \frac{\sin u}{u} = \boxed{1}$$

$u = \frac{1}{n}$.

SEQUENCE CONVERGES.

6. (10 points) For each part of this problem you are asked to give an example of a sequence with a certain property. If it is not obvious that your sequence satisfies the property, be sure to explain. If is not possible to give such an example, explain why.

(a) Give an example of a divergent sequence with a convergent subsequence.

$$\{a_n\}_{n=1}^{\infty} = \left\{1, 1, 2, \tfrac{1}{2}, 3, \tfrac{1}{3}, 4, \tfrac{1}{4}, 5, \tfrac{1}{5}, 6, \tfrac{1}{6}, \ldots\right\}$$

$a_{2n} = \frac{1}{n}$ AND $a_{2n} \to 0$ AS $n \to \infty$.

(b) Give an example of a nonconstant sequence that has limit 3/4.

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{3n}{4n+1}\right\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty} \frac{3n}{4n+1} = \lim_{n\to\infty} \frac{3}{4+\frac{1}{n}} = \frac{3}{4}$$

(c) Give an example of an unbounded sequence that converges with limit 100.

IMPOSSIBLE. CONVERGENT SEQUENCES ARE BOUNDED.

(d) Give an example of a sequence that is recursively defined.

$a_0 = 1$ AND $a_{n+1} = a_n + 2$; $n = 0, 1, 2, \ldots$

7. (5 points) Determine whether the series converges or diverges. If it converges, find its sum.

$$\sum_{n=1}^{\infty}\left(\frac{3^n+(-2)^n}{5^n}\right) = \sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n + \sum_{n=1}^{\infty}\left(-\frac{2}{5}\right)^n$$

GEOMETRIC WITH $r=\frac{3}{5}$ AND $r=-\frac{2}{5}$

$$\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n + \sum_{n=0}^{\infty}\left(-\frac{2}{5}\right)^n - 2$$

$$= \frac{1}{1-\frac{3}{5}} + \frac{1}{1+\frac{2}{5}} - 2 = \frac{5}{2}+\frac{5}{7}-2$$

SERIES CONVERGES.

$$= \boxed{\frac{17}{14}}$$

8. (5 points) The following series converges. Find it sum.

$$9n^2+3n-2 = (3n+2)(3n-1)$$

$$\sum_{n=1}^{\infty}\frac{1}{9n^2+3n-2} = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{3n-1}-\frac{1}{3n+2}\right)$$

$$\frac{1}{(3n+2)(3n-1)} = \frac{A}{3n+2}+\frac{B}{3n-1}$$

Cover up... $A=-\frac{1}{3}$

$B=\frac{1}{3}$

$$= \frac{1}{3}\left[\frac{1}{2}-\cancel{\frac{1}{5}}+\cancel{\frac{1}{5}}-\cancel{\frac{1}{8}}+\cancel{\frac{1}{8}}-\cancel{\frac{1}{11}} + \cdots\right]$$

$$= \boxed{\frac{1}{6}}$$

TELESCOPING SERIES.

9. (10 points) Consider the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}(n+1)}$.

(a) Use the result of an earlier problem on this test to show that the series converges.

LOOK AT PROBLEM #4.

$f(x)=\frac{1}{\sqrt{x}(x+1)}$ IS POSITIVE, CONT., AND DECREASING FOR $X \geq 1$.

INTEGRAL TEST APPLIES.

$\int_1^{\infty}\frac{1}{\sqrt{x}(x+1)}\,dx$ CONVERGES

$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}(n+1)}$ CONVERGES.

(b) By using an appropriate p-series, use limit comparison to show that the series converges.

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}\,n}$ IS A CONVERGENT $p=\frac{3}{2}$ SERIES.

$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n}(n+1)}}{\frac{1}{\sqrt{n}\,n}} = \lim_{n\to\infty}\frac{n}{n+1}$$

$$= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1 \Longrightarrow$$

SERIES CONVERGES BY LIMIT COMP.

10. (5 points) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ converges or diverges. If it converges, does it converge absolutely or conditionally?

THE SERIES ALTERNATES.

$a_n = \frac{1}{\ln(n+1)}$

$a_n \to 0$ AS $n \to \infty$

AND $0 \le a_{n+1} \le a_n$

BY THE AST, THE SERIES CONVERGES.

THE SERIES DOES NOT CONVERGE ABSOLUTELY BY COMPARISON WITH THE HARMONIC SERIES:

$\frac{1}{\ln(n+1)} > \frac{1}{n+1}$.

11. (5 points) Explain why the alternating series test does not apply to the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\cos(\pi n)}$.

Does the series converge or diverge?

$\cos(\pi n) = (-1)^n$ FOR $n = 1, 2, 3, \ldots$

$$\sum_{n=1}^{\infty} \frac{\cancel{(-1)^n}}{n \cancel{\cos(\pi n)}} = \sum_{n=1}^{\infty} \frac{1}{n}.$$

THE HARMONIC SERIES DIVERGES.

THE SERIES DOES NOT ALTERNATE.

12. (5 points) We will soon be able to show that $\sin(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$.

(a) Compute S_4, the partial sum through $n = 4$. Carry enough decimal digits through your computations so that you are reasonably sure that your final answer has at least 8 correct digits.

$$S_4 = \frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!}$$

$$\approx \boxed{0.8414710097}$$

(b) Use the alternating series remainder theorem to determine an upper bound on the error made in the approximation $\sin(1) \approx S_4$.

$$a_5 = \frac{1}{11!} \Rightarrow |S_4 - \sin(1)| < \boxed{\frac{1}{11!} \approx 2.5 \times 10^{-8}}$$

13. (20 points) Determine whether the series converges or diverges. Show all work when applying our tests.

(a) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

LIMIT COMPARISON WITH CONVERGENT $p = \frac{3}{2}$ SERIES, $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n^2+1} \cdot \frac{n^{3/2}}{1} = \lim_{n\to\infty} \frac{n^2}{n^2+1} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1$$

SERIES CONVERGES.

(b) $\sum_{n=1}^{\infty} \frac{\pi e^n}{3^n}$

GEOMETRIC WITH $a = \pi$ AND $r = \frac{e}{3} \approx 0.906 < 1$

SERIES CONVERGES.

(c) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+n+1}}$

n^{TH} TERM TEST

$$\lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+n+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}+\frac{1}{n^2}}} = \sqrt{1} = 1$$

$$\neq 0$$

SERIES DIVERGES.

(d) $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} \left[\ln(n+1) - \ln(n)\right]$

TELESCOPING.

$$S_n = \left[\ln(n+1) - \cancel{\ln(n)}\right] + \left[\cancel{\ln(n)} - \ln(n-1)\right] + \cdots + \left[\ln(2) - \ln(1)\right]$$

$$= \ln(n+1) - \ln(1) = \ln(n+1) \to \infty$$

AS $n \to \infty$.

SERIES DIVERGES.