Math 132 - Final Exam

December 14, 2022

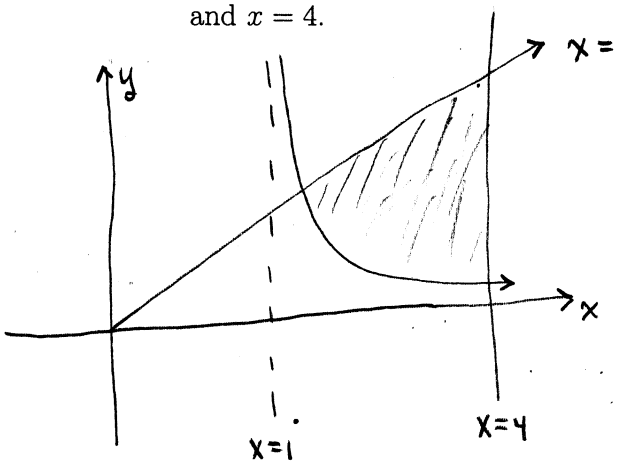
Name key

Score ________

Show all work to receive full credit. Supply explanations when necessary. Each problem is worth 5 points. You may skip one problem with no reduction of points. Please indicate which problem you wish to count for your five free points.

---

1. Find the area of the 1st quadrant region bounded by the graphs of $y = \dfrac{2}{x-1}$, $y = x$, and $x = 4$.



$$\frac{2}{x-1} = x \Rightarrow x^2 - x = 2$$

$$x^2 - x - 2 = 0 \qquad (x-2)(x+1) = 0$$

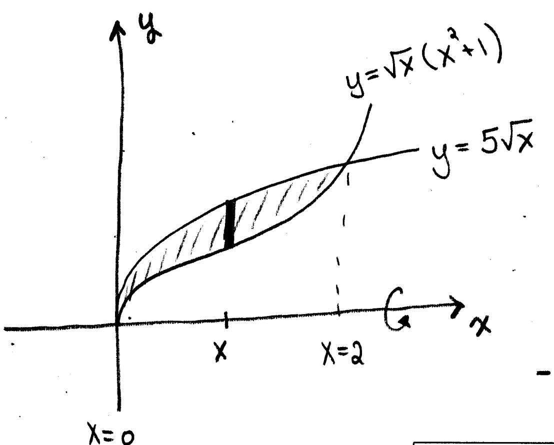
$$x = 2$$

$$\int_2^4 \left(x - \frac{2}{x-1}\right) dx = \frac{1}{2}x^2 - 2\ln|x-1| \Big|_2^4$$

$$= (8 - 2\ln 3) - (2)$$

$$\boxed{6 - 2\ln(3) \approx 3.8028}$$

2. The region bounded by the graphs of $y = \sqrt{x}(x^2+1)$ and $y = 5\sqrt{x}$ is rotated about the x-axis to form a solid. Find the volume of the solid.



WASHERS!

$$\sqrt{x}(x^2+1) = 5\sqrt{x}$$

$$\therefore \quad x = 0 \quad \text{or} \quad x^2 + 1 = 5$$

$$x = 2, \quad \cancel{x = -2}$$

$$-\pi \int_0^2 \left[\underbrace{x(x^2+1)^2}_{\text{BOTTOM}} - \underbrace{25x}_{\text{TOP}}\right] dx$$

$$u = x^2 + 1$$

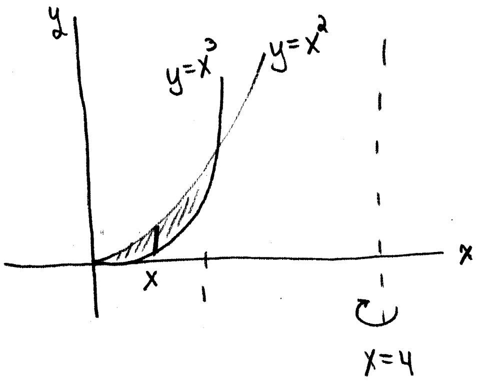
$$dx = 2x\,dx$$

$$-\pi \int_1^5 \frac{1}{2}u^2\,du + \pi \int_0^2 25x\,dx = -\frac{\pi}{6}u^3 \Big|_1^5 + \frac{25\pi}{2}x^2 \Big|_0^2$$

$$\boxed{\frac{88\pi}{3} \approx 92.1534}$$

$$-\frac{124\pi}{6} + 50\pi = \frac{176\pi}{6}$$

3. The 1st quadrant region between the graphs of $y = x^2$ and $y = x^3$ is rotated about the line $x = 4$ to form a solid. Find the volume of the solid.



CYLINDRICAL SHELLS ...

$$2\pi \int_0^1 (4-x)(x^2-x^3)\,dx = 2\pi \int_0^1 (4x^2 - 5x^3 + x^4)\,dx$$

$$= 2\pi\left(\frac{4}{3} - \frac{5}{4} + \frac{1}{5}\right)$$

$$= \frac{17\pi}{30}$$

$$\boxed{\frac{17\pi}{30} \approx 1.7802}$$

4. Evaluate the indefinite integral: $\int x^4 \ln x\,dx$

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$dv = x^4\,dx \qquad v = \frac{1}{5}x^5$

$$= \frac{1}{5}x^5 \ln x - \int \frac{1}{5}x^4\,dx$$

$$= \frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + C$$

$$\boxed{\frac{1}{5}x^5 \ln x - \frac{1}{25}x^5 + C}$$

5. Evaluate the indefinite integral: $\int \sin^3(2x)\cos^2(2x)\,dx$

$$\int \sin^2(2x)\cos^2(2x)\ \sin(2x)\,dx$$

$$= \int \left(1-\cos^2(2x)\right)\cos^2(2x)\ \sin(2x)\,dx$$

$u = \cos 2x$

$du = -2\sin 2x\,dx$

$-\frac{1}{2}du = \sin 2x\,dx$

$$-\frac{1}{2}\int (1-u^2)u^2\,du$$

$$= -\frac{1}{2}\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C$$

$$\boxed{-\frac{1}{6}\cos^3(2x) + \frac{1}{10}\cos^5(2x) + C}$$

6. Given the definite integral $\int_0^{3/5} \sqrt{9-25x^2}\,dx$, carry out the appropriate trigonometric substitution (including the integration bounds), simplify the new integrand, and stop. Do not evaluate.

Let $5x = 3\sin\theta$

$dx = \frac{3}{5}\cos\theta\,d\theta$

$x=0 \Rightarrow 3\sin\theta = 0$
$\Rightarrow \theta = 0$

$x = \frac{3}{5} \Rightarrow \sin\theta = 1$
$\Rightarrow \theta = \frac{\pi}{2}$

$$\int_0^{\pi/2} \sqrt{9-9\sin^2\theta}\left(\frac{3}{5}\right)\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} (3\cos\theta)\left(\frac{3}{5}\right)\cos\theta\,d\theta$$

$$\boxed{\frac{9}{5}\int_0^{\pi/2} \cos^2\theta\,d\theta}$$

7. Determine the partial fraction decomposition of $\dfrac{-2x+3}{(x+1)^2}$.

$$\frac{-2x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

$$-2x+3 = A(x+1) + B$$

$x=-1$: $5 = B$

$x=0$: $3 = A + B \Rightarrow A = -2$

$$\boxed{\frac{-2}{x+1} + \frac{5}{(x+1)^2}}$$

8. Use the trapezoid rule with four subintervals ($n = 4$) to approximate $\displaystyle\int_1^2 \sin(x^2)\,dx$.

$h = \dfrac{2-1}{4} = 0.25$, $f(x) = \sin(x^2)$

$$T = \frac{h}{2}\left[f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2)\right]$$

$$= 0.474845811\ldots$$

$$\boxed{\int_1^2 \sin(x^2)\,dx \approx 0.4748}$$

9. Consider the integral $\int_2^\infty \frac{dx}{\sqrt{x-2}\,(x-1)}$. Explain why the integral is improper. Then, in the box below, rewrite the improper integral with limits. DO NOT EVALUATE.

① INTEGRATION INTERVAL IS UNBOUNDED.

- AND -

② THE INTEGRAND HAS AN INFINITE DISCONT. AT $X = 2$ (THE LOWER BOUND)

$$\boxed{\lim_{c \to 2^+} \int_c^3 \frac{dx}{\sqrt{x-2}\,(x-1)} + \lim_{d \to \infty} \int_3^d \frac{dx}{\sqrt{x-2}\,(x-1)}}$$

10. Determine whether the series converges or diverges. If it converges, find its sum.

$$\sqrt{\frac{3}{\pi}} + \sqrt{\frac{9}{\pi^2}} + \sqrt{\frac{27}{\pi^3}} + \sqrt{\frac{81}{\pi^4}} + \sqrt{\frac{243}{\pi^5}} + \cdots$$

$r = \sqrt{\frac{3}{\pi}} \Rightarrow r + r^2 + r^3 + \cdots$

GEOMETRIC WITH $r = \sqrt{\frac{3}{\pi}} < 1 \Rightarrow$ CONVERGES.

$$r + r^2 + r^3 + \cdots = \frac{1}{1-r} - 1$$

GEO w/ $r < 1$ CONVERGES.	Sum $= \frac{1}{1-\sqrt{\frac{3}{\pi}}} - 1 \approx$	42.87

11. Determine whether the series converges or diverges: $\sum_{n=0}^{\infty} \tan^{-1} \sqrt{n}$

$$\lim_{n \to \infty} \tan^{-1} \sqrt{n} = \frac{\pi}{2} \neq 0$$

SERIES DIVERGES BY n^{TH} TERM TEST

12. Determine whether the series converges or diverges: $\sum_{n=1}^{\infty} \sin \frac{1}{n}$

COMPARE WITH THE HARMONIC SERIES, $\sum_{n=1}^{\infty} \frac{1}{n}$, WHICH DIVERGES.

$$\lim_{n \to \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1 \Rightarrow$$

SERIES DIVERGES BY LIMIT COMPARISON.

13. Suppose ρ is a positive constant. Find all values of ρ for which the following series converges. Is the convergence absolute or conditional?

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+\rho}$$

FOR ANY $\rho > 0$, $\frac{1}{n+\rho}$ IS DECREASING AND $\lim_{n\to\infty} \frac{1}{n+\rho} = 0$.

SERIES CONVERGES BY AST FOR ANY $\rho > 0$.

CONVERGENCE IS CONDITIONAL.

TO SHOW NOT ABSOLUTE, LIMIT COMPARE WITH $\sum \frac{1}{n}$ → $\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+\rho}} = \lim_{n\to\infty} \frac{n+\rho}{n} = 1$

$\boxed{\text{CONDITIONAL CONVERGENCE FOR ANY } \rho > 0.}$

14. Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x+3)^n}{4^n n}$.

RATIO TEST:

$$\lim_{n\to\infty} \left| \frac{(x+3)^{n+1}}{4^{n+1}(n+1)} \cdot \frac{4^n n}{(x+3)^n} \right| = \lim_{n\to\infty} \frac{|x+3|}{4} \frac{n}{n+1} = \frac{|x+3|}{4} \underbrace{\lim_{n\to\infty} \frac{n}{n+1}}_{1}$$

$$= \frac{|x+3|}{4}$$

$$\frac{|x+3|}{4} < 1 \Downarrow |x+3| < 4$$

$\boxed{\text{RADIUS OF CON.} = 4}$

15. Use the fact the $f(x) = \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$ to find a power series for $\dfrac{1}{(1-x)^3}$. (Hint: Think about $f''(x)$.)

$f'(x) = \dfrac{1}{(1-x)^2}$, $f''(x) = \dfrac{2}{(1-x)^3}$

WE NEED THE POWER SERIES FOR $\frac{1}{2} f''(x)$.

$$\frac{1}{2} f''(x) = \frac{1}{2} \sum_{n=2}^{\infty} n(n-1) x^{n-2}$$

$$\boxed{\frac{1}{(1-x)^3} = \frac{1}{2} \sum_{n=2}^{\infty} n(n-1) x^{n-2}, \quad -1 < x < 1}$$

16. Use a 3rd Taylor polynomial for $f(x) = \ln x$ centered at $x = 1$ to approximate $\ln(1.2)$.

$f(x) = \ln x \qquad f(1) = 0$

$f'(x) = \dfrac{1}{x} \qquad f'(1) = 1$

$f''(x) = -\dfrac{1}{x^2} \qquad f''(1) = -1$

$f'''(x) = \dfrac{2}{x^3} \qquad f'''(1) = 2$

$$P_3(x) = 0 + 1(x-1) + \frac{(-1)}{2}(x-1)^2 + \frac{2}{6}(x-1)^3$$

$$P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

$$\ln(1.2) \approx P_3(1.2) = 0.2 - \frac{1}{2}(0.2)^2 + \frac{1}{3}(0.2)^3$$
$$= 0.181\overline{3}$$

$$\boxed{P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3, \quad P_3(1.2) = 0.181\overline{3}}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

17. Determine the Maclaurin series for $f(x) = e^{-10x}$.

$f(x) = e^{-10x}, \quad f(0) = 1$

$f'(x) = -10e^{-10x}, \quad f'(0) = -10$

$f''(x) = 100e^{-10x}, \quad f''(0) = 100$

$\vdots$

$f^{(n)}(0) = (-1)^n 10^n$

MAC. SERIES IS

$$\sum_{n=0}^{\infty} \frac{(-1)^n 10^n}{n!} x^n$$

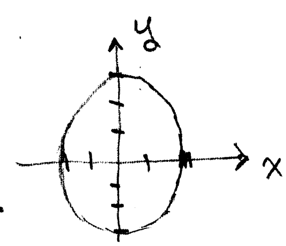
$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 10^n}{n!} x^n}$$

18. Eliminate the parameter θ to obtain an equation in x and y. Describe the graph of the resulting equation.

$$x = 2\cos\theta, \qquad y = 3\sin\theta$$

$\frac{x}{2} = \cos\theta \quad \frac{y}{3} = \sin\theta \quad \Rightarrow \quad \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \cos^2\theta + \sin^2\theta$

$$\underbrace{\frac{x^2}{4} + \frac{y^2}{9} = 1}_{\text{ELLIPSE}}$$

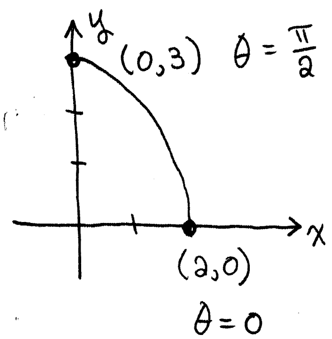


$\boxed{\frac{x^2}{4} + \frac{y^2}{9} = 1}$ THE GRAPH IS AN ELLIPSE.

19. Given the set of parametric equations from the last problem,

$$x = 2\cos\theta, \qquad y = 3\sin\theta, \qquad \frac{dx}{d\theta} = -2\sin\theta, \quad \frac{dy}{d\theta} = 3\cos\theta$$

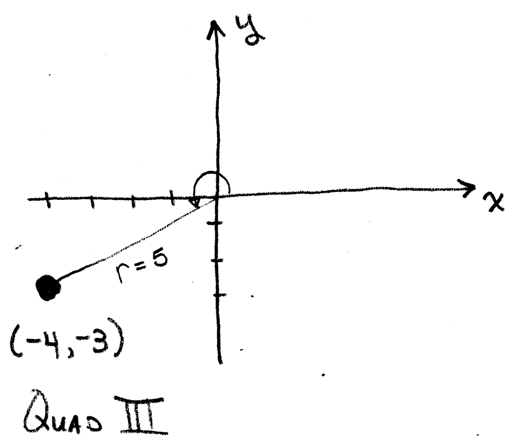
set up the definite integral that gives the length of the portion of the graph that lies in the 1st quadrant. DO NOT EVALUATE.



$$\text{ARC LENGTH} = \int_0^{\pi/2} \sqrt{(-2\sin\theta)^2 + (3\cos\theta)^2}\, d\theta$$

$$\boxed{\int_0^{\pi/2} \sqrt{4\sin^2\theta + 9\cos^2\theta}\, d\theta}$$

20. Convert the point $(x, y) = (-4, -3)$ to polar coordinates.



$$r^2 = (-4)^2 + (-3)^2 = 25$$

$$r = 5$$

$$\theta = \pi + \tan^{-1}\left(\frac{3}{4}\right)$$

$$\approx 3.78509$$

$$\approx 216.87^\circ$$

$$\boxed{(5, 3.785) \text{ OR } (5, 216.87^\circ)}$$