Math 132 - Test 1

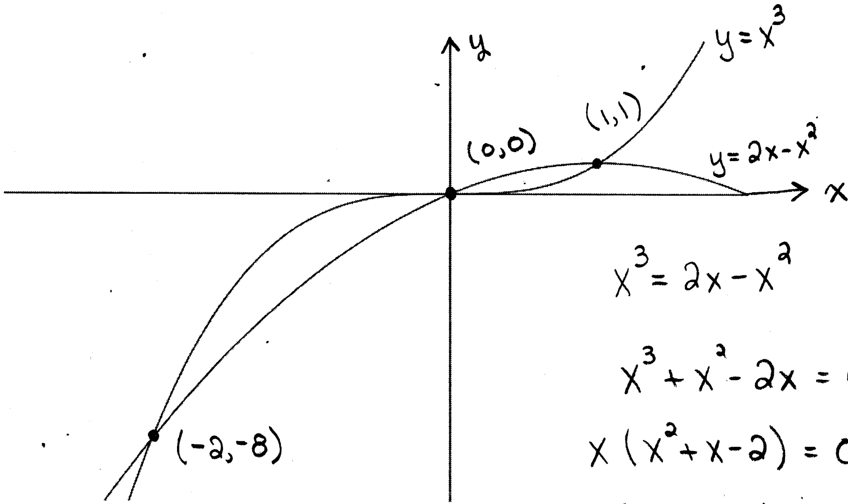
February 20, 2020

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. Unless otherwise indicated, evaluate all integrals by hand. (However, you may check your work with your calculator.)

1. (15 points) Consider the region between the graphs of $y = x^3$ and $y = 2x - x^2$. The graphs are shown below.



$$x^3 = 2x - x^2$$

$$x^3 + x^2 - 2x = 0$$

$$x(x^2 + x - 2) = 0$$

$$x(x+2)(x-1) = 0$$

(a) Find the x-coordinates of the points at which the graphs intersect.

$x = 0, \ x = -2, \ x = 1$

(b) Find the area of the combined regions enclosed by the graphs.

$$\int_{-2}^{0} x^3 - (2x - x^2)\,dx + \int_0^1 (2x - x^2) - x^3\,dx$$

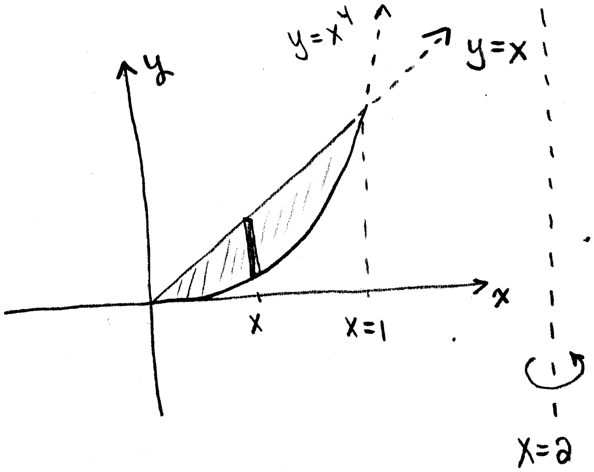
$$\frac{1}{4}x^4 - x^2 + \frac{1}{3}x^3 \Big|_{-2}^{0} + x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \Big|_0^1$$

$$= \left[0 - \left(\frac{16}{4} - 4 - \frac{8}{3}\right)\right] + \left[\left(1 - \frac{1}{3} - \frac{1}{4}\right) - 0\right]$$

$$= \frac{8}{3} + \frac{5}{12} = \boxed{\frac{37}{12}}$$

$x = x^4 \Rightarrow x = 0,\ x = 1$

2. (15 points) The bounded region between the graphs of $y = x$ and $y = x^4$ is rotated about the line $x = 2$ to form a solid. Find the volume of the solid.



Shells...

$$\text{Volume} = 2\pi \int_0^1 (2-x)(x-x^4)\,dx$$

$$2\pi \int_0^1 (2x - x^2 - 2x^4 + x^5)\,dx$$

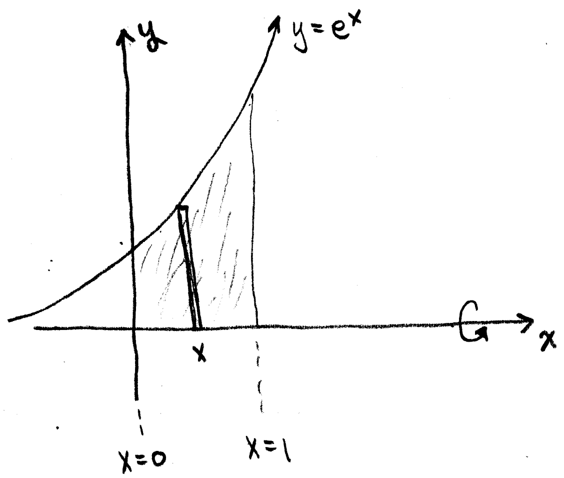
$$= 2\pi \left(x^2 - \tfrac{1}{3}x^3 - \tfrac{2}{5}x^5 + \tfrac{1}{6}x^6 \right)\Big|_0^1$$

$$= 2\pi \left(1 - \tfrac{1}{3} - \tfrac{2}{5} + \tfrac{1}{6} \right)$$

$$= 2\pi \left(\frac{13}{30} \right)$$

$$= \boxed{\frac{13\pi}{15}}$$

3. (10 points) The region bounded by the graphs of $y = e^x$, $y = 0$, $x = 0$, and $x = 1$ is rotated about the x-axis to form a solid. Find the volume of the solid.



$$\pi \int_0^1 (e^x)^2\,dx$$

$$= \pi \int_0^1 e^{2x}\,dx$$

$u = 2x$

$du = 2\,dx$

$\frac{1}{2}du = dx$

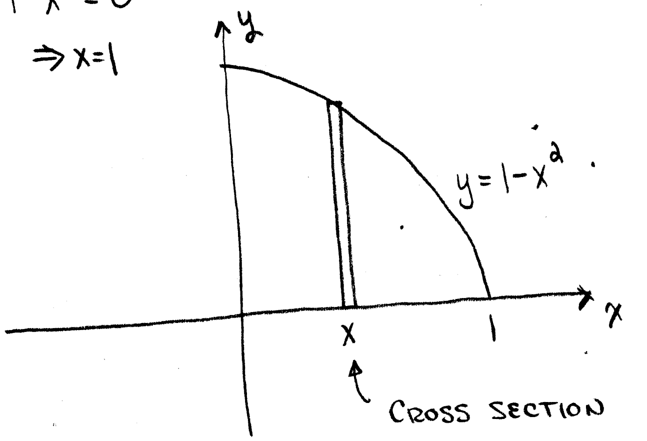
$$\frac{\pi}{2} \int_0^2 e^u\,du = \frac{\pi}{2} e^u \Big|_0^2$$

$$= \boxed{\frac{\pi}{2}\left(e^2 - 1\right)} \approx 10.036$$

4. (10 points) The base of a solid is the 1st quadrant region in the xy-plane that is under the graph of $y = 1 - x^2$. The cross sections of the solid that are perpendicular to the x-axis are squares. Find the volume of the solid.

$1 - x^2 = 0$

$\Rightarrow x = 1$



CROSS SECTION HERE IS A SQUARE OF AREA $(1-x^2)^2$

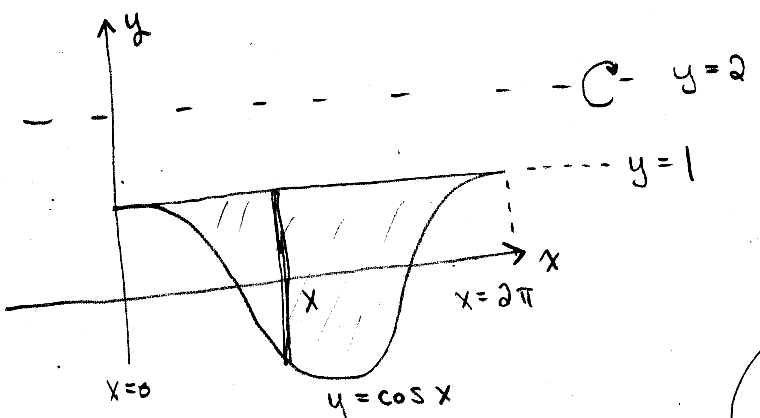
$$\text{VOLUME} = \int_0^1 (1-x^2)^2\,dx$$

$$= \int_0^1 (1 - 2x^2 + x^4)\,dx$$

$$= x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \Big|_0^1$$

$$= 1 - \frac{2}{3} + \frac{1}{5} = \boxed{\frac{8}{15}}$$

5. (6 points) The region bounded by the graphs of $y = 1$ and $y = cosx$ on the interval from $x = 0$ to $x = 2\pi$ is rotated about the line $y = 2$. Set up the definite integral that gives the volume of the region. Do not evaluate the integral.



WASHERS...

$$\boxed{\pi \int_0^{2\pi} \left[\underbrace{(2 - \cos x)^2}_{\text{OUTSIDE RADIUS}} - \underbrace{(2-1)^2}_{\text{INSIDE RADIUS}} \right] dx}$$

6. (6 points) A particle moves along the graph of $y = \ln x$ from the point $(1, 0)$ to the point $(e, 1)$. Find the distance traveled by the particle. (You may use your calculator to evaluate the required integral.)

$$\text{ARC LENGTH} = \int_1^e \sqrt{1+\left(\frac{1}{x}\right)^2}\, dx$$

$$\frac{d}{dx} \ln x = \frac{1}{x} \longrightarrow$$

$$= \int_1^e \sqrt{1+\frac{1}{x^2}}\, dx \approx 2.0035$$

7. (8 points) A thin rod lies along the x-axis extending from $x = 3$ to $x = 8$. The density of the rod at x is given by $\rho(x) = 2x - 5$. Find the center of mass of the rod. (You may use your calculator to evaluate the required integral(s).)

$$\frac{\text{Moment}}{\text{Mass}} = \frac{\int_3^8 x(2x-5)\, dx}{\int_3^8 (2x-5)\, dx} = \frac{\frac{1115}{6}}{30} = \frac{1115}{180} = \frac{223}{36} \approx 6.19$$

8. (8 points) A weightless rope is used to lift a leaky bucket of water 20 feet straight up to the top of a roof. The bucket initially weighs 38 lb. It leaks at a constant rate so that by the time it reaches the roof, it has lost 28 lb. Find the work done in lifting the bucket. (You may use your calculator to evaluate the required integral.)

BUCKET WEIGHS $w = 38$ WHEN $y = 0$

AND $w = 10.$ WHEN $y = 20$

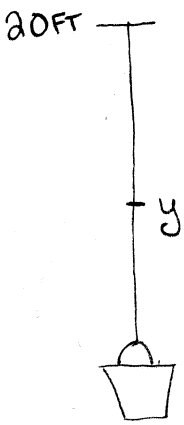
LEAKS AT CONSTANT RATE

$\Rightarrow$ w IS A LINEAR FUNCTION OF y.

INTERCEPT = $(0, 38)$

SLOPE = $\frac{-28}{20}$

$w = -\frac{28}{20} y + 38$



WEIGHT AT y

$= -\frac{28}{20} y + 38$

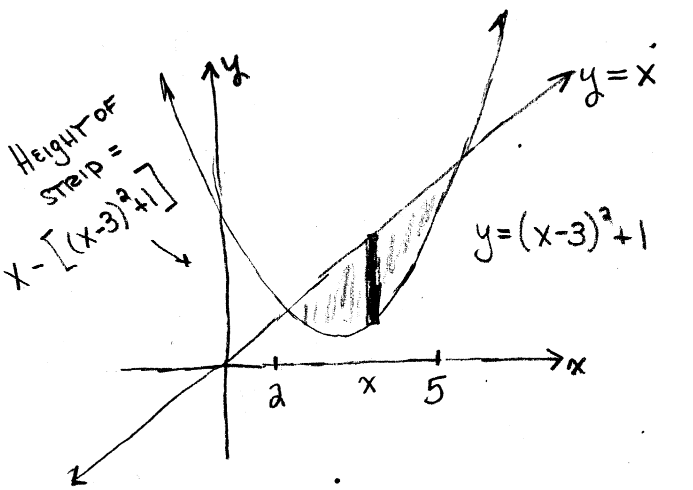
MOVES LITTLE DISTANCE dy

$$\text{WORK} = \int_0^{20} \left(-\frac{28}{20} y + 38\right) dy$$

$$= \boxed{480 \text{ ft-lb}}$$

You must work individually on the following problems. They are due by 7:35am on Tuesday, February 25.

9. (14 points) A thin plate is bounded by the graphs of $y = x$ and $y = (x-3)^2 + 1$. The density of the plate at the point (x, y) is given by $\rho(x) = \sqrt{x} - 1$. Find the center of mass of the plate. (You may use your calculator to evaluate the required integrals.)



$$X = (X-3)^2 + 1$$
$$X = X^2 - 6X + 10$$
$$X^2 - 7X + 10 = 0$$
$$(X-2)(X-5) = 0$$
$$X = 2, \quad X = 5$$

$$dm = \rho(x)\left[x - (x-3)^2 - 1\right]dx$$

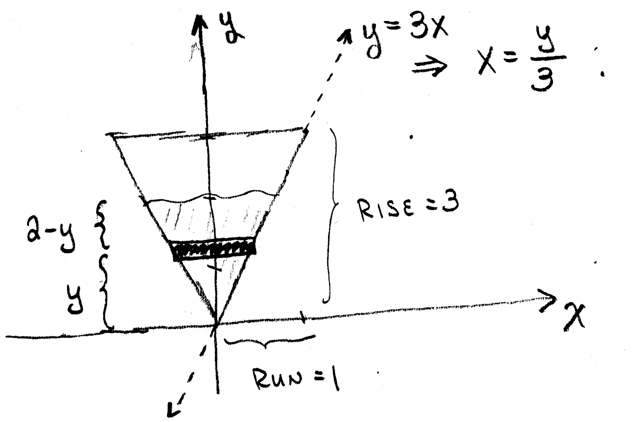
$$\text{Mass} = M = \int_2^5 (\sqrt{x}-1)\left(x - (x-3)^2 - 1\right) dx$$
$$= 3.879066$$

$$M_y = \int_2^5 x(\sqrt{x}-1)\left(x-(x-3)^2-1\right) dx$$
$$= 14.123515$$

$$M_x = \int_2^5 \left(\frac{x + (x-3)^2 + 1}{2}\right)(\sqrt{x}-1)\left(x-(x-3)^2-1\right) dx$$
$$= 10.622007$$

$$\text{C.M.} = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) \approx (3.64, 2.74)$$

10. (8 points) A metal plate has the shape of an isosceles triangle with base length 2 feet and altitude 3 feet. The plate will be used point-down (i.e., the base on top) as the end of a trough that will hold untreated sewage weighing 97 lbs/ft^3. Find the fluid force on the plate when the sewage in the trough is 2 feet deep. (Use your calculator to evaluate the required integral.)



DEPTH OF STRIP

LENGTH OF STRIP

$$\text{Fluid Force} = \int_0^2 97\,\underbrace{(2-y)}_{\text{depth of strip}}\,\underbrace{(2)\left(\tfrac{y}{3}\right)}_{\text{length of strip}}\,dy$$

$$= \frac{776}{9}\text{ lbs} \approx \boxed{86.2\text{ lbs}}$$