Math 132 - Test 3
April 16, 2020

Name key

Score ________

Show all work to receive full credit. Supply explanations when necessary. You must work individually on this test. Submit your test in Blackboard no later than Monday, April 20, at 9pm.

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1. (5 points) In a single complete sentence, clearly explain why the integral is improper. Then write it as the limit of a proper integral. Do not evaluate.

$$\int_8^{12} \frac{x-12}{\sqrt{x-8}}\,dx$$

THE INTEGRAND, $f(x) = \dfrac{x-12}{\sqrt{x-8}}$, HAS AN INFINITE DISCONTINUITY AT $x=8$.

$$\int_8^{12} \frac{x-12}{\sqrt{x-8}}\,dx = \lim_{t \to 8^+} \int_t^{12} \frac{x-12}{\sqrt{x-8}}\,dx.$$

2. (5 points) Use the trapezoid rule with four subintervals ($n = 4$) to approximate the value of the following integral. When you round, use at least 6 decimal places.

$$\int_1^2 \sin\left(\frac{1}{x}\right)\,dx$$

$h = \dfrac{2-1}{4} = \dfrac{1}{4} = 0.25$

$x_0 = 1$

$x_1 = 1.25 = \frac{5}{4}$

$x_2 = 1.5 = \frac{6}{4}$

$x_3 = 1.75 = \frac{7}{4}$

$x_4 = 2$

TRAP. RULE ESTIMATE

$$= \frac{1}{8}\left(\sin(1) + 2\sin\left(\tfrac{4}{5}\right) + 2\sin\left(\tfrac{4}{6}\right) + 2\sin\left(\tfrac{4}{7}\right) + \sin\left(\tfrac{1}{2}\right)\right)$$

$$\approx \boxed{0.634252}$$

3. (5 points) Write as the limit of a proper integral and evaluate: $\displaystyle\int_0^{\infty} \frac{4x}{x^2+1}\,dx$

SUB FIRST...

$u = x^2 + 1$

$du = 2x\,dx$

$x = 0 \Rightarrow u = 1$

$x = \infty \Rightarrow u = \infty$

$$\int_1^{\infty} \frac{2}{u}\,du = \lim_{t\to\infty} \int_1^{t} \frac{2}{u}\,du$$

$$= \lim_{t\to\infty} \left(2\ln u \Big|_1^t \right)$$

$$= \lim_{t\to\infty} \left(2\ln t \right) = \infty$$

$\boxed{\text{DIVERGES.}}$

4. (5 points) Consider the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_1 = 2$, $a_2 = 4$, and for $n > 2$, $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$. Write the first six terms of the sequence. Based on your terms, do you think the sequence converges or diverges?

$a_1 = 2$

$a_2 = 4$

$a_3 = \frac{2+4}{2} = 3$

$a_4 = \frac{3+4}{2} = 3.5$

$a_5 = \frac{3.5+3}{2} = 3.25$

$a_6 = \frac{3.25+3.5}{2} = 3.375$

$\{2, 4, 3, 3.5, 3.25, 3.375, \ldots\}$

NOT A LOT OF EVIDENCE, BUT BASED ON THE PATTERNS, IT LOOKS LIKE IT CONVERGES TO A NUMBER BETWEEN 3.25 & 3.5.

5. (5 points) Find the limit of the sequence whose nth term is $a_n = \dfrac{(n-1)^2}{(2n+1)^2}$.

LET $f(x) = \dfrac{(x-1)^2}{(2x+1)^2}$

$$\lim_{x\to\infty} \frac{(x-1)^2}{(2x+1)^2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\left(1-\frac{1}{x}\right)^2}{\left(2+\frac{1}{x}\right)^2} = \frac{1^2}{2^2} \Rightarrow \boxed{a_n \to \frac{1}{4}}$$

COULD ALSO USE L'HÔPITAL'S RULE.

6. (5 points) What does it mean for a series to converge?

A SERIES CONVERGES IF AND ONLY IF ITS SEQUENCE OF PARTIAL SUMS CONVERGES.

7. (5 points) The series $\sum_{n=0}^{\infty} \frac{2}{5^n}$ is a special kind of series. What is its name? State whether it converges or diverges (and how you know). If it converges, find its sum.

THE SERIES IS GEOMETRIC WITH $a = 2$ AND $r = \frac{1}{5}$

SINCE $|r| < 1$, THE SERIES CONVERGES AND

THE SUM IS $\frac{2}{1-\frac{1}{5}} = \boxed{\frac{5}{2}}$

8. (5 points) The series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$ is a telescoping series. Find a formula for the nth partial sum.

$S_1 = \left(1 - \frac{1}{3}\right)$

$S_2 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right)$

$S_3 = \left(1 - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right)$

$S_4 = \left(1 + \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right)$

LOOKS LIKE

$S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}$

EVERYTHING ELSE WILL CANCEL.

9. (10 points) Consider the series $\sum_{n=1}^{\infty} \frac{n}{2n^2+3}$.

(a) Apply the nth term test for divergence and describe the conclusion of the test.

$$\lim_{n\to\infty} \frac{n}{2n^2+3} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{2+\frac{3}{n^2}} = \frac{0}{2}$$

LIMIT IS ZERO. THERE IS NO CONCLUSION. THE NTH TERM TEST TELLS US NOTHING.

(b) Apply the limit comparison test by comparing the series with the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$. What is your conclusion?

$$\lim_{n\to\infty} \frac{\frac{n}{2n^2+3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{2n^2+3} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{2+\frac{3}{n^2}} = \frac{1}{2}$$

SINCE THE HARMONIC SERIES DIVERGES, THE ORIGINAL SERIES DIVERGES.

10. (5 points) Give an example of a convergent p-series and a divergent p-series. Say how you know and which is which.

CONVERGENT

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$p = 2 > 1$

DIVERGENT

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$p = \frac{1}{2} \leq 1$

11. (5 points) Steve was asked to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{3^n - n}$. He used the direct comparison test to compare his series with the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{3^n}$. What is wrong with Steve's approach?

$$\frac{1}{3^n - n} > \frac{1}{3^n}$$

SMALLER DENOM MAKES FRACTION GREATER!

THE ORIGINAL SERIES IS GREATER THAN A CONVERGENT SERIES.

THERE IS NO CONCLUSION FROM DIRECT COMPARISON.

12. (5 points) The integral test cannot be used to test the series $\sum_{n=0}^{\infty} e^{-n} \sin n$. Why not?

$$f(x) = e^{-x} \sin x$$

IS NOT A POSITIVE, DECREASING FUNCTION.

13. (5 points) Use the alternating series test to determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ converges or diverges.

$$\lim_{n \to \infty} \frac{1}{\sqrt[3]{n}} = 0$$

AND, FOR ANY POSITIVE INTEGER n,

$$\frac{1}{\sqrt[3]{n+1}} < \frac{1}{\sqrt[3]{n}}$$

SO THE TERMS $a_n = \frac{1}{\sqrt[3]{n}}$ ARE DECREASING.

$\left\{ \frac{1}{\sqrt[3]{n}} \right\}$ IS A DECREASING SEQUENCE WITH LIMIT ZERO.

$\Rightarrow$ SERIES CONVERGES BY AST.

14. (5 points) Does the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{5/4}}$ converges conditionally or absolutely? How do you know?

$$\left|\frac{(-1)^{n-1}}{n^{5/4}}\right| = \frac{1}{n^{5/4}}, \quad \sum_{n=1}^{\infty} \frac{1}{n^{5/4}} \text{ IS A CONVERGENT } p = \frac{5}{4} \text{ SERIES.}$$

SINCE THE SERIES OF ABS. VALUES CONVERGES, THE SERIES CONVERGES ABSOLUTELY.

15. (10 points) It was first discovered by Indian mathematicians in the 1300's that

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}.$$

(a) Use your calculator (or computer) to compute S_{20}, the 20th partial sum of the series. Round your final answer to 6 decimal places.

$$S_{20} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots - \frac{1}{39}$$

$$= \frac{129049485078524}{166966608033225} \approx 0.772906.$$

(b) Use the alternating series remainder theorem to determine an upper bound on the error made in using the approximation $\frac{\pi}{4} \approx S_{20}$.

$$\left|\frac{\pi}{4} - S_{20}\right| < \frac{1}{2(21)-1} = \frac{1}{41} \approx 0.024$$

THE ERROR IS LESS THAN 0.024

16. (5 points) Think about the series $\sum_{n=1}^{\infty} \frac{1}{\pi^n}$. Tamara says that the series converges, and Jon says that the series converges absolutely. Who is correct? And why?

BOTH ARE CORRECT. SINCE THE TERMS OF THE SERIES ARE ALL POSITIVE, CONVERGENCE AND ABSOLUTE CONVERGENCE ARE THE SAME THING!

17. (5 points) Looking back that series above, $\sum_{n=1}^{\infty} \frac{1}{\pi^n}$, you should recognize that it is a geometric series. Explain what is wrong with using the formula

$$\sum_{n=1}^{\infty} \frac{1}{\pi^n} = \frac{1}{1-\pi}.$$

THERE ARE TWO SERIOUS MISTAKES.

① THE SERIES IS GEOMETRIC WITH $a = 1$ AND $r = \frac{1}{\pi}$ (NOT π).

② THE FORMULA BEING APPLIED IS FOR SERIES STARTING WITH INDEX $n = 0$ (NOT $n = 1$)

WHAT IS TRUE IS THAT

$$\sum_{n=0}^{\infty} \frac{1}{\pi^n} = \frac{1}{1-\frac{1}{\pi}} = \frac{\pi}{\pi - 1}.$$

18. (5 points) As briefly as possible, tell the difference between a sequence and a series.

A SEQUENCE IS AN ORDERED <u>LIST</u> OF NUMBERS WHEREAS A SERIES IS THE <u>SUM</u> OF THE TERMS OF A SEQUENCE. THE SUM MAY OR MAY NOT MAKE SENSE, BUT THE LIST ALWAYS DOES.