Math 132 - Test 2

March 10, 2021

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. Unless otherwise indicated, evaluate all integrals by hand. (However, you may check your work with your calculator.)

1. (5 points) Use the definitions of the hyperbolic functions (in terms of exponential functions) to prove the following identity:

$$\boxed{\cosh^2(x) + \sinh^2(x)} = \cosh(2x)$$

$$\left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4} + \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$= \frac{2e^{2x} + 2e^{-2x}}{4} = \frac{e^{2x} + e^{-2x}}{2}$$

$$= \cosh(2x) \checkmark$$

2. (5 points) Use the identity above and the differentiation rules for the hyperbolic functions to derive an identity for $\sinh(2x)$.

$$\frac{d}{dx}\cosh^2 x + \frac{d}{dx}\sinh^2 x = \frac{d}{dx}\cosh 2x$$

$$2\cosh x \sinh x + 2\sinh x \cosh x = 2\sinh 2x$$

$$4\sinh x \cosh x = 2\sinh 2x$$

$$\boxed{2\sinh x \cosh x = \sinh 2x}$$

3. (5 points) Using the definitions of the hyperbolic functions (in terms of exponential functions) and the quotient rule, show that $\frac{d}{dx}[\operatorname{csch} x] = -\operatorname{csch} x \coth x$.

$$\frac{d}{dx} \operatorname{CSCH} X = \frac{d}{dx} \frac{2}{e^x - e^{-x}} = \frac{(e^x - e^{-x})(0) - (2)(e^x + e^{-x})}{(e^x - e^{-x})^2}$$

$$= \frac{-2}{e^x - e^{-x}} \cdot \frac{e^x + e^{-x}}{e^x - e^{-x}} = -\operatorname{CSCH} X \ \operatorname{COTH} X \quad ✓$$

4. (5 points) Let $f(x) = \tanh^{-1}(\cos(x))$. Use differentiation rules to find $f'(x)$.

$$f'(x) = \frac{1}{1 - \cos^2 x} \cdot (-\sin x) = \frac{-\sin x}{1 - \cos^2 x}$$

$$= \frac{-1}{\sin x} = -\csc x$$

5. (8 points) Evaluate the indefinite integral: $\int t \sin(t^2) \cos(t^2)\, dt$

LET $u = t^2$

$du = 2t\, dt$

$\frac{1}{2} du = t\, dt$

$$\frac{1}{2}\int \sin u \cos u \, du$$

LET $w = \sin u$

$dw = \cos u \, du$

$$\frac{1}{2}\int w \, dw = \frac{1}{4} w^2 + C$$

$$= \boxed{\frac{1}{4} \sin^2(t^2) + C}$$

6. (6 points) Use a substitution to evaluate the indefinite integral: $\displaystyle\int \frac{dx}{x(\ln x)^2}$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$\int \frac{1}{u^2}\,du = \int u^{-2}\,du$$

$$= -u^{-1} + C$$

$$= \boxed{-\frac{1}{\ln x} + C}$$

7. (10 points) Evaluate the definite integral: $\displaystyle\int_1^2 x^3 \ln x\,dx$

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$dv = x^3\,dx \qquad v = \frac{1}{4}x^4$

$$= \frac{1}{4}x^4 \ln x \Big|_1^2 - \int_1^2 \frac{1}{4}x^3\,dx$$

$$= \left[4\ln 2 - 0\right] - \left[\frac{1}{16}x^4 \Big|_1^2\right]$$

$$= 4\ln 2 - 1 + \frac{1}{16} = \boxed{4\ln 2 - \frac{15}{16}}$$

8. (8 points) Use integration by parts to evaluate the indefinite integral: $\int \sin x \cos x \, dx$

$u = \sin x \qquad du = \cos x \, dx$

$dv = \cos x \, dx \qquad v = \sin x$

$$\int \sin x \cos x \, dx = \sin^2 x - \underbrace{\int \sin x \cos x \, dx}_{\text{ADD TO BOTH SIDES.}}$$

$$2 \int \sin x \cos x \, dx = \sin^2 x$$

$$\longrightarrow \int \sin x \cos x \, dx = \boxed{\frac{1}{2} \sin^2 x + C}$$

9. (12 points) Evaluate the definite integral: $\int_0^{\pi/2} x^2 \sin x \, dx$

SIGNS	U AND DERIVS	dv/dx AND ANTIS
+	x^2	$\sin x$
−	$2x$	$-\cos x$
+	2	$-\sin x$
−	0	$\cos x$

$$\left(-x^2 \cos x + 2x \sin x + 2 \cos x\right)\Big|_0^{\pi/2}$$

$$= \left[0 + \pi + 0\right] - \left[0 + 0 + 2\right]$$

$$= \boxed{\pi - 2}$$

10. (8 points) Evaluate the indefinite integral: $\int \tan x \sec^3 x \, dx$

$$\int \sec^2 x \cdot \sec x \tan x \, dx$$

$$u = \sec x$$

$$du = \sec x \tan x \, dx$$

$$\int u^2 \, du = \frac{1}{3} u^3 + C$$

$$= \boxed{\frac{1}{3} \sec^3 x + C}$$

11. (8 points) In the following integral, carry out the appropriate trigonometric substitution, simplify the integrand, and then stop. Do not evaluate the new integral.

$$\int \frac{dx}{\sqrt{1+9x^2}}$$

Let $3x = \tan \theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$3\,dx = \sec^2 \theta \, d\theta$$

$$dx = \frac{1}{3} \sec^2 \theta \, d\theta$$

$$\frac{1}{3} \int \frac{\sec^2 \theta \, d\theta}{\underbrace{\sqrt{1+\tan^2 \theta}}} = \frac{1}{3} \int \frac{\sec^2 \theta}{\sqrt{\sec^2 \theta}} \, d\theta = \frac{1}{3} \int \sec \theta \cdot d\theta$$

Stop.

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\sqrt{\sec^2 \theta} = \sec \theta$$

For θ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

12. (8 points) Evaluate the indefinite integral: $\int 4\cos^2 x\,dx$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\int 4\cos^2 x\,dx = 2\cdot\int (1+\cos 2x)\,dx$$

$$= 2\left(x + \tfrac{1}{2}\sin 2x\right) + C = 2x + \sin 2x + C$$

13. (6 points) Use a product-to-sum formula to evaluate the definite integral: $\int_0^\pi \sin(3x)\sin(5x)\,dx$

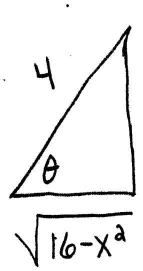
$$\sin 3x \sin 5x = \frac{1}{2}\left[\cos 2x - \cos 8x\right]$$

$$\frac{1}{2}\int_0^\pi (\cos 2x - \cos 8x)\,dx = \frac{1}{2}\left[\frac{1}{2}\sin 2x - \frac{1}{8}\sin 8x\right]_0^\pi$$

$$= \boxed{0}$$

14. (6 points) After making the trigonometric substitution $x = 4\sin\theta$, you evaluated an integral and obtained $3\theta + \cos^2\theta + C$. Resubstitute and write your result in terms of the variable x.

$$\sin\theta = \frac{x}{4}$$



$$\cos\theta = \frac{\sqrt{16-x^2}}{4}$$

$$3\theta + \cos^2\theta + C =$$

$$3\sin^{-1}\left(\frac{x}{4}\right) + \left(\frac{\sqrt{16-x^2}}{4}\right)^2 + C$$

$$= 3\sin^{-1}\left(\frac{x}{4}\right) + \frac{16-x^2}{16} + C$$