Math 132 - Test 3
April 14, 2021

Name key

Score

Show all work to receive full credit. Supply explanations where necessary. Unless otherwise indicated, evaluate all integrals by hand. (However, you may check your work with your calculator.)

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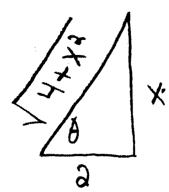
1. (10 points) Use a trig substitution to evaluate the indefinite integral: $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

$x = 2\tan\theta, \ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$dx = 2\sec^2\theta \, d\theta$



$\sec\theta = \dfrac{\sqrt{4+x^2}}{2}$

$\tan\theta = \dfrac{x}{2}$

$$\int \frac{2\sec^2\theta \, d\theta}{\sqrt{4+4\tan^2\theta}}$$

$$= \int \frac{2\sec^2\theta \, d\theta}{\underbrace{\sqrt{4\sec^2\theta}}} = \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$\sqrt{4\sec^2\theta} = 2|\sec\theta|$

$= 2\sec\theta$

$$= \boxed{\ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C}$$

2. (5 points) Write the form of the partial fraction decomposition, but DO NOT SOLVE for the undetermined coefficients.

$$\frac{x^2+3x-7}{x(3x+5)^2(x^2+7)}$$

$$\frac{x^2+3x-7}{x(3x+5)^2(x^2+7)} = \boxed{\frac{A}{x} + \frac{B}{3x+5} + \frac{C}{(3x+5)^2} + \frac{Dx+E}{x^2+7}}$$

3. (11 points) Use partial fractions to evaluate the indefinite integral.

$$\int \frac{3x}{x^2 + 2x - 8}\, dx$$

$$\frac{3x}{x^2+2x-8} = \frac{3x}{(x+4)(x-2)} = \frac{A}{x+4} + \frac{B}{x-2}$$

$$3x = A(x-2) + B(x+4)$$

$$x = 2: \quad 6 = 6B \Rightarrow B = 1$$

$$x = -4: \quad -12 = -6A \Rightarrow A = 2$$

$$\int \left(\frac{2}{x+4} + \frac{1}{x-2} \right) dx = 2 \ln|x+4| + \ln|x-2| + C$$

4. (4 points) Steve thought it might be wise to use a partial fraction decomposition to simplify the integral $\int \frac{x+5}{(x^2+1)^2}\, dx$. What do you think about Steve's strategy?

THE STRATEGY IS NO GOOD!

THE INTEGRAND, $\frac{x+5}{(x^2+1)^2}$, IS ALREADY DECOMPOSED! IT HAS THE FORM

$$\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}, \text{ WHERE}$$

$$A = 0,\ B = 0,\ C = 1,\ D = 5.$$

5. (6 points) Use the trapezoid rule with $n = 5$ to approximate $\int_1^2 \frac{dx}{x}$.

$$h = \frac{2-1}{5} = \frac{1}{5} = 0.2$$

$x_0 = 1$ $\quad$ $x_3 = 1.6$

$x_1 = 1.2$ $\quad$ $x_4 = 1.8$

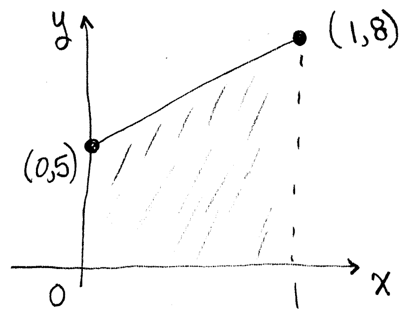
$x_2 = 1.4$ $\quad$ $x_5 = 2$

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{0.2}{2}\left[\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right] = 0.6956349$$

6. (3 points) If you used the trapezoid rule with $n = 1$ to approximate $\int_0^1 (3x+5)\,dx$, you would get the exact value of the integral. Why do you think that is so?

$y = 3x+5$ IS A LINEAR FUNCTION.

THE REGION UNDER ITS GRAPH IS A TRAPEZOID.



TRAPEZOID RULE GIVES AREA OF TRAPEZOID.

7. (11 points) Write as a limit and evaluate: $\int_{-\infty}^0 xe^x\,dx$

$$\lim_{t\to-\infty} \int_t^0 xe^x\,dx = \lim_{t\to-\infty} \left[xe^x - e^x\right]_t^0$$

$$= -1 - \lim_{t\to-\infty}\left[te^t - e^t\right]$$

$$= -1 + \underbrace{\lim_{t\to-\infty} e^t(1-t)}$$

+	x	e^x
−	1	e^x
+	0	e^x

$$\lim_{t\to-\infty}\frac{1-t}{e^{-t}} = \lim_{t\to-\infty}\frac{-1}{-e^{-t}} = 0$$

L'HÔPITAL'S RULE

VALUE OF THE INTEGRAL IS -1

8. (8 points) Write as a limit and evaluate: $\displaystyle\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$

$$\lim_{t\to 4^-} \int_0^t \frac{1}{\sqrt{4-x}}\,dx = \lim_{t\to 4^-} \Big[-2\sqrt{4-x}\Big]_0^t$$

$$= \lim_{t\to 4^-} \Big[-2\sqrt{4-t} + 4\Big]$$

$$= 0 + \boxed{4}$$

$$\int \frac{1}{\sqrt{4-x}}\,dx = -\int u^{-1/2}\,du$$

$u = 4-x$

$du = -dx$

$$= -2u^{1/2} + C$$

$$= -2\sqrt{4-x} + C$$

9. (4 points) Explain why the following integral is improper and describe how you would evaluate it. DO NOT EVALUATE.

$$\int_{-2}^{2} \frac{1}{(1+x)^2}\,dx$$

THE INTEGRAND HAS AN INFINITE DISCONTINUITY AT $X = -1$.

SPLIT THE INTEGRAL: $\displaystyle\int_{-2}^{-1} \frac{1}{(1+x)^2}\,dx + \int_{-1}^{2} \frac{1}{(1+x)^2}\,dx$.

NOW WRITE EACH AS A LIMIT AND EVALUATE.

10. (5 points) Write the first four terms of the sequence $\left\{\dfrac{n^2}{e^n}\right\}_{n=1}^{\infty}$. Then find the limit of the sequence (if it exists).

$$\left\{\frac{n^2}{e^n}\right\} = \left\{\frac{1}{e}, \frac{4}{e^2}, \frac{9}{e^3}, \frac{16}{e^4}, \ldots\right\}$$

$$\lim_{n\to\infty} \frac{n^2}{e^n} = \underbrace{\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x}}_{\text{L'HÔPITAL'S RULE}} = \underbrace{\lim_{x\to\infty} \frac{2}{e^x}}_{\text{L'HÔPITAL'S RULE}} = \boxed{0}$$

11. (5 points) Consider the sequence $\{a_n\}_{n=1}^{\infty}$, where $a_1 = 2$ and $a_{n+1} = a_n + 3$ for $n \geq 1$. Write the first four terms of the sequence and find an explicit formula for a_n.

$$\boxed{\begin{aligned} a_1 &= 2 \\ a_2 &= 2+3 = 5 \\ a_3 &= 5+3 = 8 \\ a_4 &= 8+3 = 11 \end{aligned}}$$

$$a_5 = 11+3 = 14$$

$$\{a_n\} = \{2, 5, 8, 11, 14, 17, \ldots\}$$

$$a_n = 2 + (n-1)3$$

OR

$$\boxed{a_n = 3n-1}$$

12. (8 points) Use the integral test to determine whether $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges or diverges.

$$f(x) = \frac{1}{x^2+1}, \quad x \geq 0$$

f IS PRETTY CLEARLY POSITIVE, CONTINUOUS, AND DECREASING FOR $x \geq 0$.

THE INTEGRAL TEST APPLIES!

$$\int_0^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{x^2+1}\,dx$$

$$= \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} 0 \right)$$

$$= \lim_{t\to\infty} \tan^{-1} t = \frac{\pi}{2}$$

INTEGRAL CONVERGES $\Rightarrow$ $\boxed{\text{SERIES CONVERGES.}}$

13. (20 points) Determine whether each series converges or diverges. Show worok or explain. If the series converges, find its sum.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ p-SERIES WITH $p = \frac{1}{3} < 1$.

$\boxed{\text{SERIES DIVERGES.}}$

(b) $\sum_{n=1}^{\infty} \left(\frac{3}{n} - \frac{3}{n+2}\right) = \left(\frac{3}{1} - \cancel{\frac{3}{3}}\right) + \left(\frac{3}{2} - \cancel{\frac{3}{4}}\right) + \left(\cancel{\frac{3}{3}} - \cancel{\frac{3}{5}}\right)$

$+ \left(\cancel{\frac{3}{4}} - \frac{3}{6}\right) + \left(\cancel{\frac{3}{5}} - \frac{3}{7}\right) + \cdots$

FOR $n > 1$, $S_n = 3 + \frac{3}{2} - \frac{3}{n+1} - \frac{3}{n+2}$ $\boxed{S_n \rightarrow \frac{9}{2}}$

(c) $\sum_{n=0}^{\infty} \frac{7e^n}{\pi^n} = \sum_{n=0}^{\infty} 7\left(\frac{e}{\pi}\right)^n$ GEOMETRIC w/ $r = \frac{e}{\pi} < 1$

SERIES CONVERGES.

$\boxed{S = \frac{7}{1 - \frac{e}{\pi}} = \frac{7\pi}{\pi - e} \approx 51.95}$

(d) $\sum_{n=1}^{\infty} \frac{n^2}{n^2+n}$

$$\lim_{n\to\infty} \frac{n^2}{n^2+n} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

$\boxed{\text{SERIES DIVERGES BY } n^{\text{TH}} \text{ TERM TEST}}$