Math 132 - Final Exam B

May 12, 2021

Name Key

Score ______

Show all work to receive full credit. Each problem is worth 5 points. Place your final answer in the box provided.

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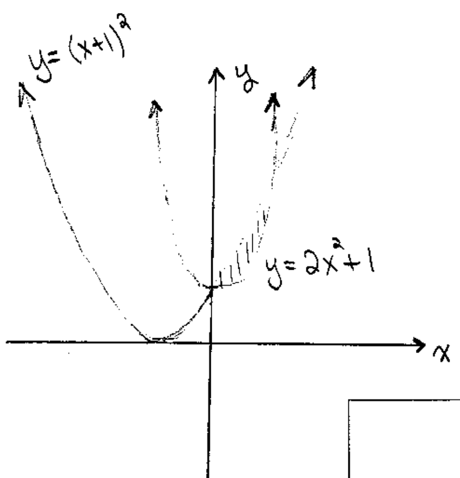
1. Find the area of the bounded region between the graphs of $y = 2x^2 + 1$ and $\underbrace{y = x^2 + 2x + 1}_{y=(x+1)^2}$.

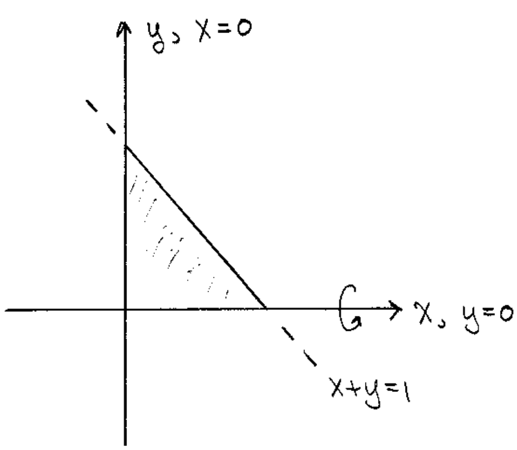


$$2x^2+1 = x^2+2x+1$$
$$x^2 - 2x = 0$$
$$x(x-2) = 0$$
$$x=0,\ x=2$$

$$\text{Area} = \int_0^2 (x^2+2x+1)-(2x^2+1)\,dx = \int_0^2 (-x^2+2x)\,dx$$
$$= -\frac{1}{3}x^3 + x^2 \Big|_0^2 = -\frac{8}{3}+4 = \boxed{\frac{4}{3}}$$

$$\boxed{4/3}$$

2. The region bounded by the graphs of $x + y = 1$, $x = 0$, and $y = 0$ is rotated about the x-axis. Find the volume of the solid that is generated.



$$\text{Volume} = \pi\int_0^1 (1-x)^2\,dx = \pi\int_0^1 (1-2x+x^2)\,dx$$
$$= \pi\left[x - x^2 + \frac{1}{3}x^3 \Big|_0^1\right] = \frac{\pi}{3}$$

$$\boxed{\pi/3}$$

3. Write $y = \operatorname{sech} x$ in terms of exponential functions. Then use your result to find the slope of the line tangent to the graph of $y = \operatorname{sech} x$ at the point where $x = 1$. Round your final answer to four decimal places.

$$y = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\frac{dy}{dx} = \frac{(e^x+e^{-x})(0) - 2(e^x - e^{-x})}{(e^x + e^{-x})^2} = \frac{-2(e^x - e^{-x})}{(e^x+e^{-x})^2}$$

$$m = \left.\frac{dy}{dx}\right|_{x=1} = \frac{-2(e-\frac{1}{e})}{(e+\frac{1}{e})^2} \approx -0.4936$$

$$\boxed{-0.4936}$$

4. Evaluate the definite integral: $\displaystyle\int_1^2 x^3 \ln x \, dx$
Write your final answer in exact form.

LIATE

$$\int x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 dx$$

$u = \ln x \quad dv = x^3 dx$

$du = \frac{1}{x} dx \quad v = \frac{1}{4}x^4$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

$$\int_1^2 x^3 \ln x \, dx = \left.\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 \right|_1^2$$

$$= (4\ln 2 - 1) - \left(0 - \frac{1}{16}\right)$$

$$= 4\ln 2 - \frac{15}{16}$$

≈ 1.8351

$$\boxed{4\ln 2 - \frac{15}{16}}$$

5. Evaluate the indefinite integral: $\int \sin^3 2x \cos^5 2x \, dx$

$$\int \sin^2 2x \cos^5 2x \sin 2x \, dx = \int (1-\cos^2 2x) \cos^5 2x \sin 2x \, dx$$

$u = \cos 2x$, $du = -2 \sin 2x \, dx$

$-\frac{1}{2} du = \sin 2x \, dx$

$$-\frac{1}{2}\int (1-u^2) u^5 \, du = -\frac{1}{2}\int (u^5 - u^7) \, du$$

$$= -\frac{1}{2}\left(\frac{u^6}{6} - \frac{u^8}{8}\right) + C$$

$$\boxed{-\frac{1}{12}\cos^6 2x + \frac{1}{16}\cos^8 2x + C}$$

6. Use a partial fraction decomposition to evaluate $\int \frac{x-1}{x^2+x} \, dx$.

$$\frac{x-1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{-1}{x} + \frac{2}{x+1}$$

$x - 1 = A(x+1) + Bx$

$x = 0 : \; -1 = A$

$x = -1 : \; -2 = -B$

$$\int \left(-\frac{1}{x} + \frac{2}{x+1}\right) dx$$

$$= -\ln|x| + 2\ln|x+1| + C$$

$$\boxed{-\ln|x| + 2\ln|x+1| + C}$$

7. Suppose the series $\sum_{n=1}^{\infty} a_n$ converges with sum $S = 3$.

(a) If S_n denotes the nth partial sum of the series, find $\lim_{n\to\infty} S_n$.

(b) Find $\lim_{n\to\infty} a_n$.

(a) IF A SERIES CONVERGES, ITS SUM IS THE LIMIT OF THE SEQUENCE OF PARTIAL SUMS : $\lim_{n\to\infty} S_n = 3$

(b) n^{TH} TERM TEST : IF $\sum a_n$ CONVERGES, THEN $\lim_{n\to\infty} a_n = 0$

(a) $\lim_{n\to\infty} S_n = 3$ (b) $\lim_{n\to\infty} a_n = 0$

8. Determine whether the series converges or diverges. If it converges, find the sum.

$$\sum_{n=1}^{\infty} \frac{7}{2^n} = \sum_{n=1}^{\infty} 7\left(\frac{1}{2}\right)^n$$

CONVERGES SINCE $|r| = \frac{1}{2} < 1$

$$\sum_{n=0}^{\infty} \frac{7}{2^n} = \frac{7}{1-\frac{1}{2}} = 14$$

$$\therefore \sum_{n=1}^{\infty} \frac{7}{2^n} = 14 - 7 = \boxed{7}$$

CONVERGES, SUM = 7

9. Use logarithm laws to rewrite as a telescoping series. Then determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \ln n - \ln(n+1)$$

$$= (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \cdots + (\ln n - \ln(n+1)) + \cdots$$

$$S_n = \ln 1 - \ln(n+1) = -\ln(n+1)$$

$$\lim_{n\to\infty} S_n = -\infty$$

$\boxed{\lim_{n\to\infty} S_n = -\infty. \text{ SERIES DIVERGES.}}$

10. Use a test to determine whether the series converges or diverges.

$$\sum_{k=1}^{\infty} \frac{3k^2+7k}{5^k\,k^2}$$

$\sum \frac{1}{5^k}$ IS GEOMETRIC WITH $r = \frac{1}{5}$. IT CONVERGES.

LET'S USE LIMIT COMPARISON WITH $\sum \frac{1}{5^k}$ ...

$$\lim_{k\to\infty} \frac{\frac{3k^2+7k}{5^k k^2}}{\frac{1}{5^k}} = \lim_{k\to\infty} \frac{(3k^2+7k)\cancel{5^k}}{\cancel{5^k}\,k^2} = \lim_{k\to\infty} \frac{3k^2+7k}{k^2} = 3$$

POS & FINITE.

$\boxed{\text{SERIES CONVERGES BY LIMIT COMPARISON.}}$

11. Determine whether the series converges conditionally, absolutely, or not at all. Show work or explain your reasoning.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt[3]{n}}$$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ IS A p-SERIES WITH $p = \frac{1}{3}$. IT DIVERGES,

SO THE SERIES CANNOT CONVERGE ABSOLUTELY.

AST...

$a_n = \frac{1}{\sqrt[3]{n}}$. THE a_n'S ARE OBVIOUSLY DECREASING WITH LIMIT ZERO. THE SERIES CONVERGES.

$\boxed{\text{CONVERGES CONDITIONALLY.}}$

12. The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges to $1/e$. Use the first six terms of the series (through $n = 5$) to approximate $1/e$ and find an upper bound on the error made in the approximation.

$$\frac{1}{e} \approx 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} = 0.3\overline{6}$$

SIX TERMS. FIRST NEGLECTED TERM IS $\frac{1}{720}$

$\boxed{\frac{1}{e} \approx 0.3\overline{6}. \quad \text{ERROR} < \frac{1}{720}}$