Math 157 - Final Exam

December 14, 2016

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) Use algebra to find the limit analytically. $\frac{0}{0}$

$$\lim_{r\to -1} \frac{3(r+2)^2+6r+3}{r+1}$$

$$= \lim_{r\to -1} \frac{3r^2+12r+12+6r+3}{r+1}$$

$$= \lim_{r\to -1} \frac{3r^2+18r+15}{r+1} = \lim_{r\to -1} \frac{3(r+1)(r+5)}{r+1}$$

$$= \lim_{r\to -1} 3(r+5) = \boxed{12}$$

2. (10 points) Let $f(x) = 1 + \overbrace{\sqrt[3]{2x+4}}^{(2x+4)^{1/3}}$. Compute $f'(x)$. Then use it to find an equation of the line tangent to the graph of f at the point where $x = 2$.

$$f'(x) = \frac{1}{3}(2x+4)^{-2/3}(2)$$

$$f'(2) = \frac{1}{3}(8)^{-2/3}(2) = \frac{2}{12} = \frac{1}{6}$$

$$m = \frac{1}{6}$$

Point: $x = 2,\ y = f(2) = 3$

$$y - 3 = \frac{1}{6}(x-2)$$

or

$$y = \frac{1}{6}x + \frac{8}{3}$$

3. (6 points) Let $f(x) = 2/x^2$. Use at least four small intervals to estimate $f'(2)$.

$[1.9, 2.1]$ ... $\frac{f(2.1)-f(1.9)}{2.1-1.9} = -0.5025094063\ldots$

$[1.99, 2.01]$ ... $\frac{f(2.01)-f(1.99)}{2.01-1.99} = -0.5000250009\ldots$

$[1.999, 2.001]$ ... $= -0.50000025\ldots$

$[1.9999, 2.0001]$ ... $= -0.500000025\ldots$

Looks like $f'(2) = -\frac{1}{2}$

4. (8 points) A bacteria culture is growing in such a way that after t hours there are $P(t) = 145e^{0.2t}$ bacteria. Compute $P(6)$ and $P'(6)$. Using units, explain what each of these values represents.

$P(6) = 481.4 \approx 481$ BACTERIA ← # OF BACTERIA AFTER 6 HRS.

$P'(t) = 0.2(145)e^{0.2t}$

$P'(6) = 96.3$ BACTERIA/HR ← GROWTH RATE AFTER 6 HRS

5. (10 points) Find the global (absolute) extreme values of the function $f(x) = 2x^3 - 9x^2 - 60x + 10$ on the interval $[-4, 4]$.

$f'(x) = 6x^2 - 18x - 60$

$= 6(x^2 - 3x - 10)$

$= 6(x-5)(x+2) = 0$

$\cancel{x=5}$ or $x=-2$

CRIT PT: $x = -2$

END PTS: $x = -4$, $x = 4$

$f(-4) = -22$

$f(4) = -246$ ← ABS MIN

$f(-2) = 78$ ← ABS MAX

6. (10 points) Use the second derivative to determine whether the graph of $y = x^2e^{-x}$ is concave up or down at the point where $x = 1$.

$y = f(x) = x^2e^{-x}$

$f'(x) = 2xe^{-x} - x^2e^{-x}$

$f''(x) = 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x}$

$f(1) = 2e^{-1} - 2e^{-1} - 2e^{-1} + e^{-1} = -e^{-1} \approx -0.368 < 0$

$\Rightarrow$ GRAPH IS CD

7. (20 points) Consider the function $f(x) = 7 + 2x^3 - x^4$.

(a) Find all critical points of f.

$$f'(x) = 6x^2 - 4x^3$$
$$= 2x^2(3-2x) = 0$$
$$\Rightarrow \quad x = 0 \text{ or } x = \frac{3}{2}.$$

(b) Find open intervals on which f is increasing/decreasing.

Signs of f': + (before 0), + (between 0 and $\frac{3}{2}$), − (after $\frac{3}{2}$)

f: INC, 0, INC, $\frac{3}{2}$, DEC

$\boxed{\text{Increasing on } (-\infty, 0) \cup (0, \tfrac{3}{2}), \text{ Decreasing on } (\tfrac{3}{2}, \infty)}$

(c) Find all local (relative) extreme values of f.

$$f\left(\frac{3}{2}\right) = \frac{139}{16} = 8.6875$$

is a rel max.

(d) Find open intervals on which the graph of f is concave up/down.

$$f''(x) = 12x - 12x^2$$
$$= 12x(1-x) = 0$$
$$\Rightarrow x = 0 \text{ or } x = 1$$

Signs of f'': − (before 0), + (between 0 and 1), − (after 1)

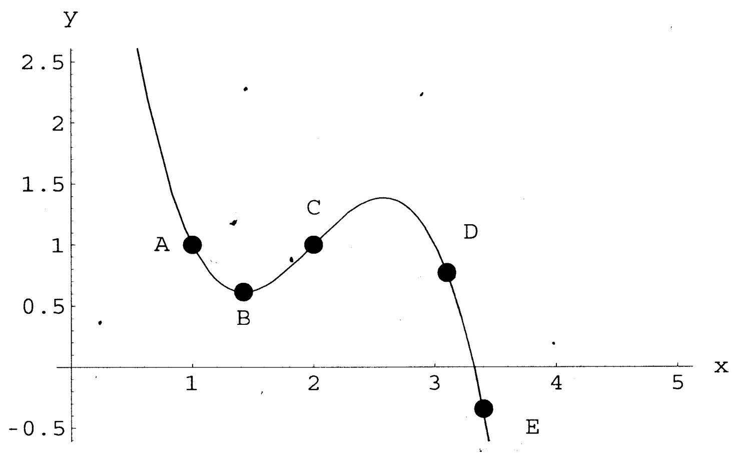
f: CD, 0, CU, 1, CD

$\boxed{\text{Concave down on } (-\infty, 0) \cup (1, \infty) \text{ and concave up on } (0, 1)}$

8. (8 points) Evaluate the indefinite integral: $\displaystyle\int \left(\frac{1}{x^2} - 7x^4 + e^{-2x} + 2x^{-1}\right) dx.$

$$\int \left(x^{-2} - 7x^4 + e^{-2x} + \frac{2}{x}\right) dx = \boxed{-\frac{1}{x} - \frac{7}{5}x^5 - \frac{1}{2}e^{-2x} + 2\ln|x| + C}$$

9. (12 points) The graph of f is shown below. For each part of this problem, find a labeled point that satisfies the given condition.



(a) $f''(x) = 0$

C (INFLECTION PT)

(b) x is a critical point of f

B $(f'(x) = 0)$

(c) $f''(x) < 0$

CONCAVE DOWN AT D OR E

(d) $f(x) < 0$

E (GRAPH BELOW X-AXIS)

(e) $f'(x) > 0$

INCREASING AT C.

(f) $f''(x) > 0$

CONCAVE UP AT A OR B

10. (12 points) An ice cream company finds that at a price of \$4.00, demand is 4000 units. For every \$0.25 decrease in price, demand increases by 200 units.

(a) Find the demand equation.

$$m = \frac{\Delta p}{\Delta q} = \frac{-0.25}{200} = -0.00125$$

Point $(4000, 4)$.

$$p - 4 = -0.00125(q - 4000)$$

or

$$\boxed{p = -0.00125q + 9}$$

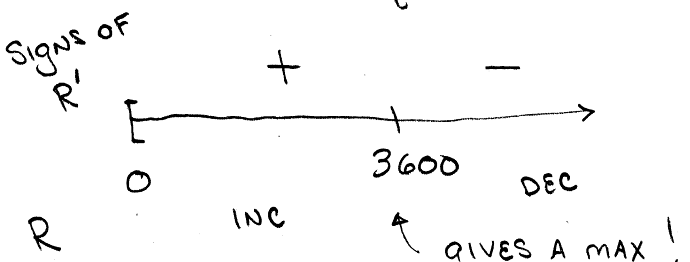
(b) Find a formula for the revenue.

$$\boxed{R = pq = -0.00125q^2 + 9q}$$

(c) Find the price and quantity that maximize revenue. (Be sure to show how you know you've found a max.)

$$R' = -0.0025q + 9 = 0$$

$$\Rightarrow q = 3600$$

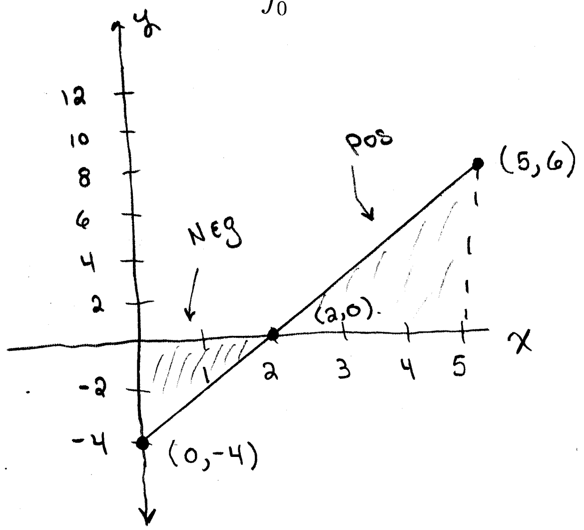


Max revenue when

$q = 3600$

$p = \$4.50$

11. (10 points) Sketch the graph of $f(x) = 2x - 4$. Then use area (not antidifferentiation) to compute $\int_0^5 f(x)\,dx$.



$$\int_0^5 f(x)\,dx = -\frac{1}{2}(2)(4) + \frac{1}{2}(3)(6)$$

$$= -4 + 9 = \boxed{5}$$

12. (8 points) Use a left sum with 5 subintervals (rectangles) of equal width to estimate $\displaystyle\int_1^2 \frac{3}{x+5}\,dx$.

$\Delta x = \frac{2-1}{5} = \frac{1}{5} = 0.2$

PARTITION : $1 < 1.2 < 1.4 < 1.6 < 1.8 < 2$

* * * * *

LEFT SUM $= 0.2\left[\frac{3}{6} + \frac{3}{6.2} + \frac{3}{6.4} + \frac{3}{6.6} + \frac{3}{6.8}\right]$

≈ 0.4697

13. (8 points) Evaluate the indefinite integral: $\displaystyle\int 4x^3\left(x^4+8\right)^5 dx$.

$u = x^4 + 8$

$du = 4x^3\,dx$

$\int u^5\,du = \frac{1}{6}u^6 + C$

$= \boxed{\frac{1}{6}\left(x^4+8\right) + C}$

14. (4 points) For each part, circle the correct conclusion.

(a) If f' is increasing, then f'' POSITIVE

i. the graph of f is concave up.

ii. the graph of f has positive slope.

(b) If $f(c) = 0$, then

i. c is a critical point of f.

ii. $(c, 0)$ is an x-intercept of the graph of f.

15. (6 points) The revenue from selling q items is $R(q) = 30q + 5$ and the total cost is $C(q) = 0.01q^3 - 0.7q^2 + 34q + 8$.

(a) Determine the profit function $P(q)$.

$P = R - C$ $\quad P(q) = [30q + 5] - [0.01q^3 - 0.7q^2 + 34q + 8]$

$\Rightarrow P(q) = -0.01q^3 + 0.7q^2 - 4q - 3$

(b) Determine the marginal profit.

$P'(q) = -0.03q^2 + 1.4q - 4$

(c) Determine the marginal profit at $q = 25$. Based on your value should you increase or decrease production in order to increase profit? Why?

$P'(25) = 12.25 > 0 \Rightarrow$ PROFIT IS INCREASING

$\Rightarrow$ INCREASE PRODUCTION.

16. (10 points) Use a definite integral (and the Fundamental Theorem of Calculus) to find the area of the region under the graph of $f(x) = 8x^3 + 6x^2 + 1$ over the interval from $x = 0$ to $x = 2$.

f IS POSITIVE ON $[0,2]$

$\Rightarrow \quad \text{AREA} = \int_0^2 (8x^3 + 6x^2 + 1)\,dx$

$= 2x^4 + 2x^3 + x \Big|_0^2 = (32 + 16 + 2) - (0 + 0 + 0)$

$= 50$