Math 171 - Test 2

October 21, 2010

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) If $g(x) = 7x^3(x - 4x^2)$, find the value of $g''(2)$.

$$g(x) = 7x^4 - 28x^5$$

$$g'(x) = 28x^3 - 140x^4$$

$$g''(x) = 84x^2 - 560x^3$$

$$g''(2) = 84(4) - 560(8) = \boxed{-4144}$$

2. (8 points) Let $f(x) = (\sin x)^4$. Find $f'(x)$. Then find an equation of the line tangent to the graph of f at the point where $x = \pi/4$.

$$f'(x) = 4\sin^3 x \cos x$$

$$f'\left(\frac{\pi}{4}\right) = 4\left(\frac{\sqrt{2}}{2}\right)^3\left(\frac{\sqrt{2}}{2}\right) = 4\left(\frac{4}{16}\right) = 1$$

Point: $x = \frac{\pi}{4}$, $y = \sin^4\left(\frac{\pi}{4}\right) = \frac{1}{4}$

Slope: $m = 1$

$$\boxed{\text{Line: } y - \frac{1}{4} = 1\left(x - \frac{\pi}{4}\right)}$$

3. (12 points) An object is launched from a tall building so that its height in feet after t seconds (measured from the ground) is given by $s(t) = -16t^2 + 64t + 512$.

(a) What is the velocity of the object after 7 seconds?

$$\text{Velocity} = v(t) = s'(t) = -32t + 64$$

$$s'(7) = -32(7) + 64 = \boxed{-160 \text{ FT/sec}}$$

(b) What is the maximum height of the object?

$$v(t) = 0 \Rightarrow -32t + 64 = 0 \Rightarrow t = 2$$

$$s(2) = -16(4) + 64(2) + 512 = \boxed{576 \text{ FT}}$$

(c) When does the object hit the ground?

$$s(t) = 0 \Rightarrow -16(t^2 - 4t - 32) = 0$$

$$\Rightarrow -16(t-8)(t+4) = 0 \Rightarrow t = \boxed{8 \text{ sec}}$$

4. (8 points) Assume that y is implicitly defined as a function of x by the following equation:

$$xy = 2x - y^3.$$

Find $\frac{dy}{dx}$ at $(1, 1)$.

$$\frac{d}{dx}(xy) = \frac{d}{dx}(2x - y^3)$$

$$x\frac{dy}{dx} + y = 2 - 3y^2\frac{dy}{dx}$$

$$x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 2 - y$$

$$\frac{dy}{dx}(x + 3y^2) = 2 - y$$

$$\frac{dy}{dx} = \frac{2-y}{x+3y^2}$$

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{2-1}{1+3} = \frac{1}{4}$$

5. (5 points) Suppose that the infected region of an injury is circular, and its radius is growing at a rate of 1.3 mm/hr. Find the rate of change of area of the infected region when the radius is 3 mm.

r = RADIUS OF REGION (mm)

A = AREA OF REGION (mm^2)

t = ELAPSED TIME (HR)

$$\frac{dr}{dt} = 1.3$$

FIND $\frac{dA}{dt}$ WHEN $r = 3$.

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

WHEN $r = 3$...

$$\frac{dA}{dt} = 2\pi(3)(1.3)$$

$$\frac{dA}{dt} = 7.8\pi$$

$$\approx \boxed{24.5 \text{ mm}^2/\text{HR}}$$

6. (8 points) Find the absolute maximum and minimum values of $g(x) = x^3 - 3x^2 + 1$ on the interval $[-1/2, 4]$.

$$g'(x) = 3x^2 - 6x$$

$$g'(x) = 0 \Rightarrow 3x(x-2) = 0$$

$$x = 0,\ x = 2$$

$g'(x)$ DNE NEVER.

CRIT NUMBERS ARE

$$x = 0,\ x = 2.$$

END POINTS:

$$x = -\tfrac{1}{2},\ x = 4$$

$$g(0) = 1$$

$$\boxed{g(2) = -3 \longleftarrow \text{ABS MIN}}$$

$$g(-\tfrac{1}{2}) = \tfrac{1}{8}$$

$$\boxed{g(4) = 17 \longleftarrow \text{ABS MAX}}$$

7. (15 points) Differentiate. Do not simplify.

(a) $\dfrac{d}{dw}\dfrac{\sqrt{w}}{w^2+3w+1}$

$$= \frac{(w^2+3w+1)\left(\frac{1}{2}\right)(w^{-1/2}) - (w^{1/2})(2w+3)}{(w^2+3w+1)^2}$$

(b) $\dfrac{d}{dt}(t^7-7t)\tan t$

$$= (t^7-7t)\sec^2 t + (7t^6-7)\tan t$$

(c) $\dfrac{d}{dx}\cos(x^3+x)$

$$= -\sin(x^3+x)\left[3x^2+1\right]$$

8. (6 points) Find all critical numbers of the function $f(x) = 5x^{2/3} + x^{5/3}$.

$$f'(x) = \frac{10}{3}x^{-1/3} + \frac{5}{3}x^{2/3} = \frac{5}{3}x^{-1/3}(2+x) = \frac{5(2+x)}{\sqrt[3]{x}}$$

$f'(x) = 0 \Rightarrow x = -2$, $f'(x)$ DNE $\Rightarrow x = 0$

CRIT #'S ARE $\boxed{x = 0,\ x = -2}$

9. (3 points) If $f'(5) = 7$, $f(5) = 0$, $g(1) = 5$, and $g'(1) = 3$, determine $\dfrac{d}{dx}f(g(x))$ when $x = 1$.

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

WHEN $x = 1$, $f'(g(1))g'(1) = f'(5)g'(1)$

$= (7)(3) = \boxed{21}$

10. (15 points) Consider the function $f(x) = x^4 + 4x^3 - 36x^2$.

(a) Determine $f'(x)$.

$$\boxed{f'(x) = 4x^3 + 12x^2 - 72x}$$

(b) Find all x-values for which f' is zero or not defined.

$f'(x)$ DNE NEVER.

$$f'(x) = 4x(x^2 + 3x - 18) = 4x(x+6)(x-3) = 0$$

$$\boxed{x = 0,\ x = -6,\ x = 3}$$

(c) Determine open intervals on which f is increasing/decreasing.

SIGNS OF
$f'(x) = 4x(x+6)(x-3)$

$-$		$+$		$-$		$+$
DEC	-6	INC	0	DEC	3	INC

f IS INC ON $(-6, 0) \cup (3, \infty)$

f IS DEC ON $(-\infty, -6) \cup (0, 3)$

(d) Identify all relative extreme values of f.

$f(-6) = -864$ IS A REL MIN

$f(3) = -135$ IS A REL MIN

$f(0) = 0$ IS A REL MAX

(e) Evaluate f'' at one of the critical numbers of f. How does the sign of f'' at that point support your conclusion in part (d)?

$$f''(x) = 12x^2 + 24x - 72$$

$f''(-6) = 216 > 0 \Rightarrow$ GRAPH IS CU $\Rightarrow f(-6)$ IS A REL MIN

$f''(0) = -72 < 0 \Rightarrow$ GRAPH IS CD $\Rightarrow f(0)$ IS A REL MAX

$f''(3) = 108 > 0 \Rightarrow$ GRAPH IS CU $\Rightarrow f(3)$ IS A REL MIN

11. (6 points) What do the signs of f'' say about the graph of f?

IF f'' IS POSITIVE ON (a,b), THEN THE GRAPH OF f IS CONCAVE UP ON (a,b).

IF f'' IS NEGATIVE ON (a,b), THEN THE GRAPH OF f IS CONCAVE DOWN ON (a,b).

12. (5 points) Suppose the function f satisfies the following conditions:

- f is continuous on $[0,5]$ and differentiable on $(0,5)$
- $f'(x) < 2$ for all x in $[0,5]$
- $f(0) = 0$

MEAN VALUE THM APPLIES ON $[0,5]$!

Use the Mean Value Theorem to show that $f(5) < 10$.

ACCORDING TO MVT, THERE EXISTS c IN $(0,5)$ SUCH THAT

$$f'(c) = \frac{f(5)-f(0)}{5-0} = \frac{f(5)}{5}$$

SINCE $f'(c) < 2$, WE HAVE

$$\frac{f(5)}{5} < 2 \quad \text{OR} \quad f(5) < 10. \quad ☺$$

13. (3 points) What is an inflection point?

AN INFLECTION POINT IS A POINT AT WHICH A GRAPH HAS A TANGENT LINE AND THE GRAPH CHANGES CONCAVITY.