Math 171 - 1st Final Exam

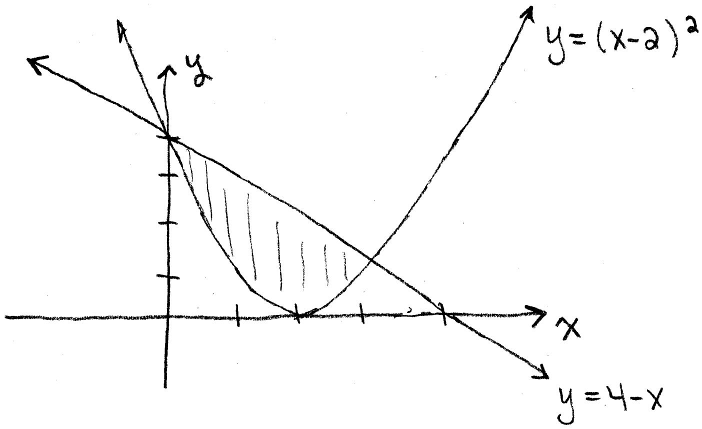
December 2, 2010

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (12 points) Find the area of the region bounded by the graphs of $\underbrace{y = x^2 - 4x + 4}_{y=(x-2)^2}$ and $y = 4 - x$.



$$x^2-4x+4 = 4-x$$

$$\Rightarrow \quad x^2-3x=0$$

$$x(x-3)=0 \Rightarrow x=0,\ x=3$$

$$\text{Area} = \int_0^3 (4-x)-(x^2-4x+4)\,dx$$

$$= \int_0^3 (3x-x^2)\,dx = \frac{3}{2}x^2 - \frac{1}{3}x^3 \Big|_0^3$$

$$= \frac{27}{2} - 9 = \boxed{\frac{9}{2}}$$

2. (8 points) Use the second derivative to determine whether the graph of

$$f(x) = \cos x + \sin 9x$$

is concave up or down at the point where $x = 5$.

$$f'(x) = -\sin x + 9\cos 9x$$

$$f''(x) = -\cos x - 81\sin 9x$$

$$f''(5) = -69.2068\ldots < 0$$

$\Rightarrow$ <u>GRAPH IS CONCAVE DOWN.</u>

3. (8 points) Suppose oil spills out from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m?

r = RADIUS OF SPILL AT TIME t

A = AREA OF SPILL AT TIME t

$A = \pi r^2$, $\quad \frac{dr}{dt} = 1 \text{ m/s}$, $\quad$ FIND $\left.\frac{dA}{dt}\right|_{r=30}$.

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\left.\frac{dA}{dt}\right|_{r=30} = 2\pi(30)(1) \approx 188.5 \text{ m}^2/\text{SEC}$$

4. (12 points) An object is moving in such a way that its height, in feet, at time t, in seconds, is given by $s(t) = -16t^2 + 42t + 20$. Determine (a) the maximum height of the object and (b) the time when the object hits the ground.

a) $s'(t) = -32t + 42$

$s'(t) = 0 \Rightarrow t = \frac{42}{32}$

$$s\left(\frac{42}{32}\right) = \frac{761}{16} = \boxed{47.5625 \text{ FT}}$$

b) $s(t) = 0$

$$\Rightarrow -16t^2 + 42t + 20 = 0 \Rightarrow t = \frac{-42 \pm \sqrt{42^2 + 4(16)(20)}}{32}$$

$\cancel{t = -0.41164}$ OR

$$\boxed{t = 3.0366 \text{ SEC}}$$

5. (15 points) Find $\dfrac{dy}{dx}$.

(a) $y = 3x^2 + 7 + \cos x$

$$\frac{dy}{dx} = 6x - \sin x$$

(b) $y = x^2 \sec x$

$$\frac{dy}{dx} = 2x \sec x + x^2 \sec x \tan x$$

(c) $y = (1 + \sin^2 x)^2$

$$\frac{dy}{dx} = 2(1 + \sin^2 x)(2 \sin x) \cos x$$

6. (8 points) Assume that y is implicitly defined as a function of x by the equation $\sin y + xy^2 = 1$. Find dy/dx.

$$\frac{d}{dx}(\sin y + xy^2) = 0$$

$$\cos y \frac{dy}{dx} + y^2 + 2xy \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(\cos y + 2xy) = -y^2$$

$$\boxed{\frac{dy}{dx} = \frac{-y^2}{\cos y + 2xy}}$$

7. (8 points) Evaluate the following definite integral: $\int_{-2}^{2} 5x^2(x^3+8)^2\,dx.$

$u = x^3 + 8$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$x = -2 \Rightarrow u = 0$

$x = 2 \Rightarrow u = 16$

$$\frac{5}{3}\int_0^{16} u^2\,du$$

$$= \frac{5}{3}\,\frac{u^3}{3}\Big|_0^{16} = \frac{5}{9}\,16^3 = \boxed{\frac{20480}{9} \approx 2275.6}$$

8. (12 points) Use the first derivative test to find open intervals on which f is increasing/decreasing.

$$f(x) = \frac{x^2+3x+1}{x+3}$$

$$f'(x) = \frac{(x+3)(2x+3) - (x^2+3x+1)(1)}{(x+3)^2}$$

$$f'(x) = \frac{2x^2+9x+9 - x^2 - 3x - 1}{(x+3)^2}$$

$$= \frac{x^2+6x+8}{(x+3)^2} = \frac{(x+2)(x+4)}{(x+3)^2}$$

$f'(x) = 0 \Rightarrow x = -2,\ x = -4$

$f'(x)$ DNE $\Rightarrow$ $x = -3$

Signs of f': $+$ | -4 | $-$ | -3 | $-$ | -2 | $+$

f inc, f dec, f dec, f inc

Increasing on

$(-\infty, -4) \cup (-2, \infty)$

Decreasing on

$(-4, -3) \cup (-3, -2)$

9. (15 points) Find each limit analytically. Use ∞, $-\infty$, or DNE if appropriate.

(a) $\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x}$ $\quad \frac{0}{0}$ $\quad$ MULT BY CONJUGATE.

$$\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x} \cdot \frac{\sqrt{x+3}+\sqrt{3}}{\sqrt{x+3}+\sqrt{3}} = \lim_{x\to 0} \frac{x}{x\left(\sqrt{x+3}+\sqrt{3}\right)} =$$

$$\lim_{x\to 0} \frac{1}{\sqrt{x+3}+\sqrt{3}} = \boxed{\frac{1}{2\sqrt{3}}}$$

(b) $\lim_{y\to 2}(2y^2-3y-2)$

$$= 2(2)^2-3(2)-2 = 8-6-2 = \boxed{0}$$

(c) $\lim_{\theta\to 2^-} \frac{\theta-3}{\theta-2}$ $\quad \frac{-1}{0}$ $\quad$ + OR − ∞ MUST CHECK.

TO LEFT OF 2, NUMERATOR IS −

DENOMINATOR IS − $\Rightarrow$ $\boxed{\text{LIMIT IS } +\infty}$

10. (6 points) Find the linearization of $f(x) = \sin 5x$ at $x = 0$.

$$L(x) = f(0) + f'(0)(x-0)$$

$f(0) = 0$

$f'(x) = 5\cos 5x \quad \rightarrow \quad \boxed{L(x) = 5x}$

$f'(0) = 5$

(2 pts extra credit) Use your linearization to justify the following limit: $\lim_{x\to 0} \frac{\sin 5x}{2x} = \frac{5}{2}$.

$$\lim_{x\to 0} \frac{\sin 5x}{2x} = \lim_{x\to 0} \frac{5x}{2x} = \lim_{x\to 0} \frac{5}{2} = \frac{5}{2}$$

11. (18 points) Consider the definite integral $\int_1^3 \frac{1}{x}\,dx$.

(a) In one or two complete sentences, explain why our formula for the antiderivative of x^n cannot be used to evaluate the definite integral.

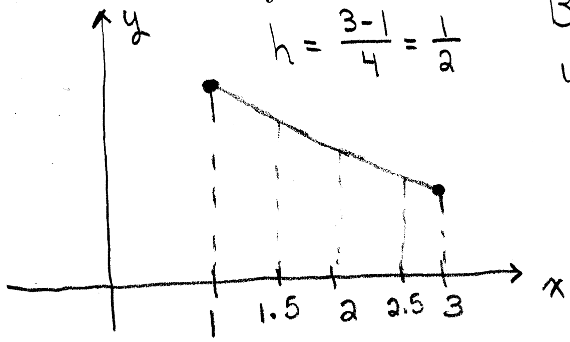
SINCE $\frac{1}{X}$ IS CONTINUOUS ON $[1,3]$, THE DEFINITE INTEGRAL EXISTS. HOWEVER, THE FUNDAMENTAL THM DOES NOT APPLY BECAUSE $\frac{1}{X} = X^{-1}$ AND IF WE USE THE POWER RULE WE'D GET $\frac{1}{0}X^0$. DIVISION BY ZERO IS NOT DEFINED.

(b) Use the trapezoid rule with four subintervals to approximate the value of the integral. Does your approximation over-estimate or under-estimate the exact value? How do you know?

BECAUSE THE GRAPH IS CONCAVE UP, THE APPROX. IS AN OVER-ESTIMATE.

$f(x) = \frac{1}{X}$

$h = \frac{3-1}{4} = \frac{1}{2}$



$$T = \frac{h}{2}\left[f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)\right]$$

$$= \frac{1}{4}\left[1 + \frac{2}{1.5} + \frac{2}{2} + \frac{2}{2.5} + \frac{1}{3}\right] = \boxed{1.11\overline{6}}$$

(c) Using four subintervals of equal length and left endpoints, compute a Riemann sum corresponding to the definite integral.

$$S = \frac{1}{2}\left[f(1) + f(1.5) + f(2) + f(2.5)\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{1.5} + \frac{1}{2} + \frac{1}{2.5}\right] = \boxed{1.28\overline{3}}$$

12. (8 points) Evaluate the following indefinite integral: $\int \left(\sqrt[3]{y^2} + \frac{1}{y^2} + \csc^2 y\right)\,dy$

$$\int \left(y^{2/3} + y^{-2} + \csc^2 y\right) dy$$

$$= \boxed{\frac{3}{5}y^{5/3} - y^{-1} - \cot y + C}$$

13. (20 points) Do any TWO of the following problems.

(a) Use the limit definition of the derivative to find $f'(x)$ if $f(x) = x - x^2$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[x+h-(x+h)^2] - [x-x^2]}{h}$$

$$= \lim_{h\to 0} \frac{x+h-x^2-2xh-h^2-x+x^2}{h} = \lim_{h\to 0} \frac{h-2xh-h^2}{h} = \lim_{h\to 0} (1-2x-h)$$

$$= \boxed{1-2x}$$

(b) State the Mean Value Theorem (MVT). Then apply the MVT to the function $f(x) = x^2$ on $[0, 3]$ and find the appropriate number c.

SUPPOSE f IS CONT ON $[a,b]$ AND DIFF ON (a,b). THEN THERE EXISTS c IN (a,b) SUCH THAT

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

$f(x) = x^2$, $f'(x) = 2x$

$$2c = \frac{3^2 - 0^2}{3-0} = 3$$

$$\boxed{c = \frac{3}{2}}$$

(c) Use Newton's method to find either one of the two points where the graph of $y = \cos x$ intersects the graph of $y = x^2$. Show each one of your estimates, rounded to the nearest ten-thousandth.

$x^2 = \cos x \Rightarrow f(x) = x^2 - \cos x$

$f'(x) = 2x + \sin x$

$$x_{N+1} = x_N - \frac{f(x_N)}{f'(x_N)}$$

$x_0 = 1$

$x_1 = 0.8382$

$x_2 = 0.8242$

$\boxed{x_3 = 0.8241}$

SAME BUT OPPOSITES FOR $x_0 = -1$

(d) Using the definition of continuity, describe three reasons why a function may fail to be continuous at a point.

f IS CONTINUOUS AT $x = c$ IF

$$\lim_{x\to c} f(x) = f(c)$$

1. f DNE AT $x = c$
2. LIMIT DNE AT $x = c$
3. LIMIT AT c $\neq$ FUNC VALUE AT c