Math 171 - 2nd Final Exam

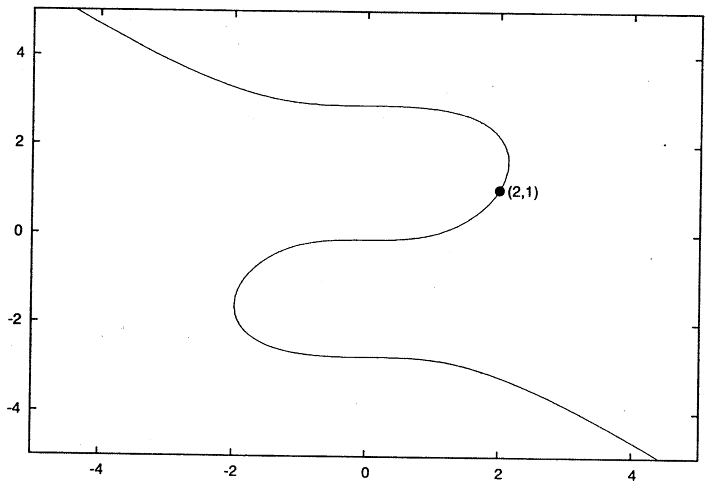
December 15, 2010

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) Find an equation of the line tangent to the graph of the equation $x^3 + y^3 = 8y + 1$ at the point $(x, y) = (2, 1)$.



$$\frac{d}{dx}\left(x^3+y^3\right) = \frac{d}{dx}\left(8y+1\right)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 8\frac{dy}{dx}$$

$$3x^2 = (8-3y^2)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3x^2}{8-3y^2}$$

$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{12}{5}$$

$$\boxed{\text{TAN LINE : } y-1 = \frac{12}{5}(x-2)}$$

2. (8 points) Find the absolute maximum and minimum values of $g(x) = \frac{x}{2} + \cos x$ on the interval $[0, 2]$.

$$g'(x) = \frac{1}{2} - \sin x$$

$$g'(x) = 0 \Rightarrow \sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6}$$

ONLY SOLUTION IN $[0,2]$.

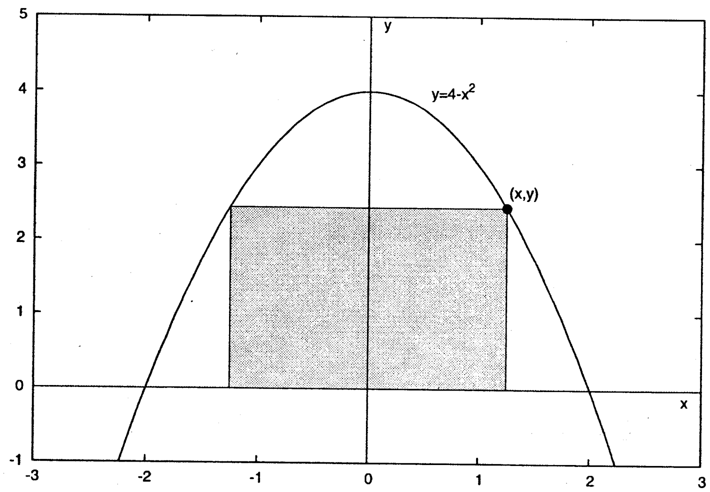
EVALUATE g AT CRIT #'s AND ENDPOINTS:

$g(0) = 1$

$g(2) = 0.583853$ ← ABS MIN

$g\left(\frac{\pi}{6}\right) = 1.12782$ ← ABS MAX

3. (12 points) A rectangle is bounded by the x-axis and the graph of $y = 4 - x^2$ (see below). Find the coordinates of the point (x, y) that maximize the area of the rectangle.



MAXIMIZE AREA

$A = 2xy$

s.t. $y = 4 - x^2$

$A(x) = 2x(4 - x^2)$, $0 \le x \le 2$

$A(x) = 8x - 2x^3$

$A'(x) = 8 - 6x^2 = 0$

$\Rightarrow x^2 = \frac{8}{6} = \frac{4}{3}$

$\Rightarrow x = \frac{2}{\sqrt{3}}$

SIGNS OF $A'(x)$

$+$ $-$

$[0 \quad \frac{2}{\sqrt{3}} \quad 2]$

$x = \frac{2}{\sqrt{3}}$ GIVES A MAX.

COORDS ARE $\left(\frac{2}{\sqrt{3}}, \frac{8}{3}\right)$

4. (7 points) Evaluate the definite integral: $\int_0^1 x^3 \cos(x^4 + 2)\, dx$.

$u = x^4 + 2$

$du = 4x^3\, dx$

$\frac{1}{4} du = x^3 dx$

$x = 0 \Rightarrow u = 2$

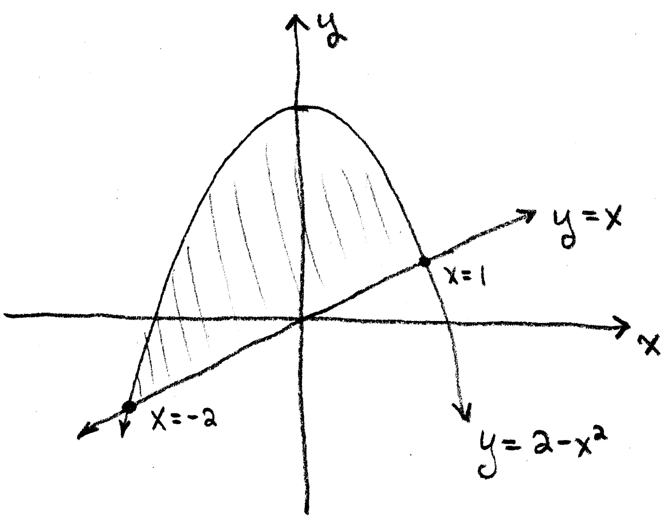
$x = 1 \Rightarrow u = 3$

$\frac{1}{4}\int_2^3 \cos u\, du$

$= \frac{1}{4} \sin u \Big|_2^3$

$= \frac{1}{4}\left[\sin 3 - \sin 2\right] \approx -0.192$

5. (11 points) Find the area of the bounded region between the graphs of $y = 2 - x^2$ and $y = x$.



$$\text{AREA} = \int_{-2}^{1} (2 - x^2 - x)\,dx$$

$$= 2x - \frac{1}{3}x^3 - \frac{1}{2}x^2 \Big|_{-2}^{1}$$

$$= \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - 2\right)$$

$$= \boxed{4.5}$$

$2 - x^2 = x$

$\Rightarrow x^2 + x - 2 = (x+2)(x-1) = 0$

$\Rightarrow x = -2,\ x = 1$

6. (10 points) A person standing at the top of the Tower of Pisa throws a small heavy object directly upward so that after t seconds the object's height in feet is given by

$$s(t) = -16t^2 + 96t + 176.$$

(a) When does the object reach its maximum height?

$s'(t) = -32t + 96$

$s'(t) = 0 \Rightarrow \boxed{t = 3 \text{ SEC}}$

(b) What is the maximum height of the object?

$s(3) = -16(9) + 96(3) + 176 = \boxed{320 \text{ FT}}$

(c) What is the velocity of the object when its height is 64 ft? What is the speed?

$-16t^2 + 96t + 176 = 64$

$0 = 16t^2 - 96t - 112 = 16(t^2 - 6t - 7) = 16(t-7)(t+1)$

$\Rightarrow t = 7 \text{ SEC}$

$$\boxed{\begin{aligned} \text{VELOCITY} &= s'(7) = -128 \text{ FT/SEC} \\ \text{SPEED} &= |s'(7)| = 128 \text{ FT/SEC} \end{aligned}}$$

7. (18 points) Consider the function $f(x) = 2x(x-3)^2$.

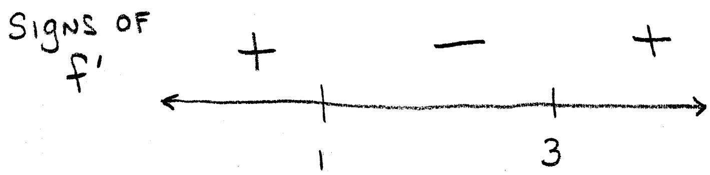
(a) Find all critical numbers of f.

$$f'(x) = 2(x-3)^2 + 4x(x-3) = (x-3)(6x-6) = 6(x-1)(x-3) = 0$$

$$\Rightarrow \boxed{X = 3 \text{ OR } X = 1}$$

$f'(x)$ DNE NEVER

(b) Find open intervals on which f is increasing/decreasing.



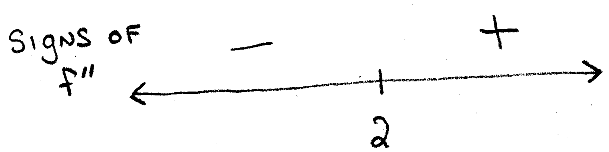
$\boxed{f \text{ IS INCREASING ON } (-\infty,1) \cup (3,\infty),\ f \text{ IS DECREASING ON } (1,3)}$

(c) Find the relative extreme values of f.

$\boxed{\begin{array}{l} f(1) = 8 \text{ IS A RELATIVE MAX} \\ f(3) = 0 \text{ IS A RELATIVE MIN} \end{array}}$

(d) Find open intervals on which the graph of f is concave up/down.

$$f''(x) = 6(x-1) + 6(x-3) = 12x - 24 = 0 \Rightarrow X = 2$$

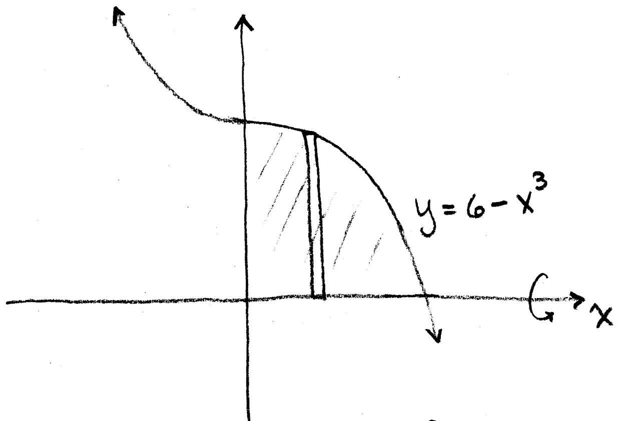


$\boxed{\begin{array}{l} \text{GRAPH IS CD ON } (-\infty,2) \\ \text{GRAPH IS CU ON } (2,\infty) \end{array}}$

(e) Find all points of inflection of the graph of f.

$(2, f(2)) = \boxed{(2,4) \text{ IS THE ONLY INFLECTION POINT}}$

8. (8 points) The **first quadrant region** under the graph $f(x) = 6 - x^3$ is rotated about the x-axis to form a solid. Find the volume of the solid. (After setting up the appropriate integral, you may use your calculator to approximate its value.)



$$\text{Volume} = \pi \int_0^{\sqrt[3]{6}} (6-x^3)^2\,dx$$

$$\approx 132.1145$$

$$6 - x^3 = 0$$
$$\Rightarrow x = \sqrt[3]{6}$$

9. (12 points) Find $\frac{dy}{dx}$. Do not simplify.

(a) $y = 3\sqrt{1-4x^2} = 3(1-4x^2)^{1/2}$

$$\frac{dy}{dx} = \frac{3}{2}(1-4x^2)^{-1/2}(-8x)$$

(b) $y = \frac{\sec x}{x^2+1}$

$$\frac{dy}{dx} = \frac{(x^2+1)(\sec x)(\tan x) - (\sec x)(2x)}{(x^2+1)^2}$$

(c) $y = x^3 \sin 2x$

$$\frac{dy}{dx} = 3x^2 \sin 2x + 2x^3 \cos 2x$$

10. (12 points) Find each limit analytically. Use ∞, $-\infty$, or DNE if appropriate.

(a) $\lim_{\theta \to 0} \frac{\sin 5\theta}{3\theta}$ $= \lim_{\theta \to 0} \frac{5}{3} \frac{\sin 5\theta}{5\theta} = \frac{5}{3} \lim_{\theta \to 0} \frac{\sin 5\theta}{5\theta}$

$= \frac{5}{3} \cdot (1) = \boxed{\frac{5}{3}}$

(b) $\lim_{y \to 4} \frac{y^2 - 3y - 4}{3y - 12}$ $= \lim_{y \to 4} \frac{(y-4)(y+1)}{3(y-4)} = \lim_{y \to 4} \frac{y+1}{3} = \boxed{\frac{5}{3}}$

(c) $\lim_{x \to -\infty} \frac{4 - 7x - 8x^2}{4x^2 + 6x + 2}$ $\cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \to -\infty} \frac{4/x^2 - 7/x - 8}{4 + 6/x + 2/x^2}$

$= \frac{0 - 0 - 8}{4 + 0 + 0} = \boxed{-2}$

11. (6 points) Find the linearization of $f(x) = \tan x$ at $x = \pi/4$.

$L(x) = f(\frac{\pi}{4}) + f'(\frac{\pi}{4})(x - \frac{\pi}{4})$

$f(\frac{\pi}{4}) = 1, \quad f'(x) = \sec^2 x$ $\longrightarrow$ $\boxed{L(x) = 1 + 2(x - \frac{\pi}{4})}$

$f'(\frac{\pi}{4}) = 2$

12. (6 points) Evaluate the indefinite integral: $\int \left(5x\sqrt{x} + \frac{2}{x^5} + \sin x\right)\, dx$

$\int (5x^{3/2} + 2x^{-5} + \sin x)\, dx$

$= \boxed{2x^{5/2} - \frac{1}{2}x^{-4} - \cos x + C}$

13. (13 points) Consider the definite integral $\int_0^1 \frac{1}{x^2+1}\,dx$.

(a) Briefly explain how we can be sure that the value of this integral is positive.

THE INTEGRAND, $f(x) = \frac{1}{x^2+1}$, IS A POSITIVE-VALUED FUNCTION.

THEREFORE, THE REGION UNDER THE GRAPH OF $f(x)$ OVER $[0,1]$ LIES IN THE 1ST QUAD $\Rightarrow$ INTEGRAL IS +.

(b) Use the trapezoid rule with $n = 4$ to approximate the value of the integral.

$N = 4 \Rightarrow h = 0.25 \qquad f(x) = \frac{1}{x^2+1}$

$$T = \frac{0.25}{2}\left[f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1)\right]$$

$$\approx \boxed{0.782794}$$

(c) Use Simpson's rule with $n = 4$ to approximate the value of the integral.

$N = 4 \Rightarrow h = 0.25 \qquad f(x) = \frac{1}{x^2+1}$

$$S = \frac{0.25}{3}\left[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)\right]$$

$$\approx \boxed{0.785392}$$

14. (5 points) What is the difference between a removable discontinuity and a nonremovable discontinuity?

A DISCONTINUITY AT $X = C$ IS REMOVABLE IF $\lim_{X \to C} f(x)$ EXISTS. IF THE LIMIT DNE AT $X = C$, THE DISCONTINUITY IS NONREMOVABLE.

15. (14 points) Do any TWO of the following problems in the space provided below.

(a) Find the average value of $f(x) = 3x^5 - x^3 + x$ on the interval $[-3, 3]$.

(b) Let $F(x) = \displaystyle\int_{x^2}^{0} \frac{\sin t}{t}\,dt$. Find $F'(x)$.

(c) The area under the graph of $y = f(x)$ on the interval $[2, 4]$ is approximated by a Riemann sum of the form

$$\sum_{k=1}^{n} (c_k^2 + c_k)\Delta x,$$

where the interval $[2, 4]$ is partitioned into n subintervals of equal width Δx and c_k is some point in the kth subinterval. Write and evaluate the definite integral that gives the exact area.

(d) Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \sqrt{x-1}$.

a) $\text{Avg} = \frac{1}{6}\int_{-3}^{3} (3x^5 - x^3 + x)\,dx = \boxed{0}$ (THE INTEGRAND IS AN ODD FUNCTION ON $[-3,3]$.)

b) $\frac{d}{dx}\int_{x^2}^{0} \frac{\sin t}{t}\,dt = -\frac{d}{dx}\int_{0}^{x^2} \frac{\sin t}{t}\,dt = -\left(\frac{\sin x^2}{x^2}\right)(2x)$

$= \boxed{-\frac{2\sin x^2}{x}}$

c) $\int_2^4 (x^2 + x)\,dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 \Big|_2^4$

$= \left[\frac{1}{3}(64) + \frac{1}{2}(16)\right] - \left[\frac{1}{3}(8) + \frac{1}{2}(4)\right] = \boxed{\frac{74}{3}}$

d) $f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h-1} - \sqrt{x-1}}{h} \cdot \frac{\sqrt{x+h-1} + \sqrt{x-1}}{\sqrt{x+h-1} + \sqrt{x-1}}$

$= \lim_{h\to 0} \frac{(x+h-1) - (x-1)}{h\left[\sqrt{x+h-1} + \sqrt{x-1}\right]} = \lim_{h\to 0} \frac{1}{\sqrt{x+h-1} + \sqrt{x-1}}$

$= \boxed{\frac{1}{2\sqrt{x-1}}}$