Math 171 - Test 2
October 18, 2012

Name $\underline{\text{key}}$

Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (6 points) A particle is moving along the graph of $x^2y = 12$ in such a way that $\frac{dx}{dt} = 8$. Find $\frac{dy}{dt}$ when $x = 2$.

$$\frac{d}{dt}(x^2y) = \frac{d}{dt}(12)$$

$$2x\frac{dx}{dt}y + x^2\frac{dy}{dt} = 0$$

WHEN $x = 2$, $(2)^2y = 12 \Rightarrow y = 3$

PLUG EVERYTHING IN ...

$$2(2)(8)(3) + (2)^2\frac{dy}{dt} = 0 \Rightarrow \boxed{\frac{dy}{dt} = -24}$$

2. (8 points) Find the critical numbers of $g(x) = x^{2/3}(x^2 - 4)$.

$$g'(x) = \frac{2}{3}x^{-1/3}(x^2-4) + x^{2/3}(2x)$$

$$= x^{-1/3}\left[\frac{2}{3}(x^2-4) + 2x^2\right] = x^{-1/3}\left[\frac{8}{3}x^2 - \frac{8}{3}\right]$$

$$g'(x) = \frac{8(x^2-1)}{3\sqrt[3]{x}}$$

$g'(x) = 0 \Rightarrow$ $x = 1$ OR $x = -1$

$g'(x)$ DNE WHEN $x = 0$

3. (12 points) An object is launched upward with an initial speed of 80 ft/s over the side of a 96-ft building.

(a) Find a formula for the object's height at time t. Use $g = 32\,\text{ft/s}^2$.

$$s(t) = -16t^2 + 80t + 96$$

(b) What is the object's average velocity over the first 3 seconds of travel?

$$\frac{s(3) - s(0)}{3-0} = \frac{192-96}{3} = 32 \text{ FT/s}$$

(c) Determine the object's velocity function.

$$v(t) = s'(t) = -32t + 80$$

(d) Determine the maximum height of the object.

$$v(t) = 0 \Rightarrow t = \frac{80}{32} = 2.5$$

$$s(2.5) = 196 \text{ FT}$$

(e) When will the object hit the ground?

$$s(t) = 0 \Rightarrow -16t^2 + 80t + 96 = 0$$

$$-16(t^2 - 5t - 6) = 0$$

$$-16(t-6)(t+1) = 0 \Rightarrow t = 6 \text{ s}$$

(f) Determine the object's acceleration function.

$$a(t) = v'(t) = -32$$

4. (8 points) Find an equation of the line tangent to the graph of $x^2y + y^3 = 2x^2$ at the point $(-1, 1)$.

$$\frac{d}{dx}\left(x^2y + y^3\right) = \frac{d}{dx}\left(2x^2\right)$$

$$x^2\frac{dy}{dx} + 2xy + 3y^2\frac{dy}{dx} = 4x$$

$$x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 4x - 2xy$$

$$\frac{dy}{dx} = \frac{4x - 2xy}{x^2 + 3y^2}$$

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{-4+2}{1+3} = -\frac{1}{2}$$

$$\boxed{y - 1 = -\frac{1}{2}(x+1)}$$

5. (9 points) An oil tanker has run aground and ruptured its hull. Leaking oil is spreading in all directions. The polluted region is circular and growing steadily at a rate $35\,\mathrm{m}^2/\mathrm{hr}$. How fast is the radius of the oil slick growing at the moment when the radius is $50\,\mathrm{m}$?

A = AREA OF REGION AT TIME t

r = RADIUS AT TIME t

$\frac{dA}{dt} = 35$ FIND $\frac{dr}{dt}$ WHEN $r = 50$.

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

$$35 = 2\pi(50)\frac{dr}{dt} \Rightarrow \boxed{\frac{dr}{dt} = \frac{35}{100\pi}\ \mathrm{m/HR}}$$

6. (20 points) Differentiate. Do not simplify.

(a) $\frac{d}{dt}\left(6t^4 + \frac{8}{t^2} - \sqrt[3]{t^2}\right) = \frac{d}{dt}\left(6t^4 + 8t^{-2} - t^{2/3}\right)$

$= \boxed{24t^3 - 16t^{-3} - \frac{2}{3}t^{-1/3}}$

(b) $\frac{d}{dx}\left[5 \cdot \tan(x) \cdot \sec(x)\right] = \boxed{5\tan(x)\sec(x)\tan(x) + 5\sec^3(x)}$

(c) $\frac{d}{dr}\csc(2\pi r^2) = \boxed{-\csc(2\pi r^2)\cot(2\pi r^2)(4\pi r)}$

(d) $\frac{d}{dx}\left[x^2\sqrt{x^6+7}\right] = \frac{d}{dx}\left(x^2(x^6+7)^{1/2}\right)$

$= \boxed{x^2\left(\frac{1}{2}\right)(x^6+7)^{-1/2}(6x^5) + 2x(x^6+7)^{1/2}}$

7. (8 points) The following table gives information about the functions f and g.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	-7	0	4	2
1	-8	-1	3	5
2	-7	4	DNE	DNE
3	2	15	1	5

(a) Use the information to find the derivative of $f(x)g(x)$ (product) at the point where $x = 1$.

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + f'(x)g(x)$$

$$f(1)g'(1) + f'(1)g(1) = (-8)(5) + (-1)(3)$$
$$= \boxed{-43}$$

(b) Use the information to find the derivative of $f(g(x))$ (composition) at the point where $x = 1$.

$$\frac{d}{dx} f(g(x)) = f'(g(x))\,g'(x)$$

$$f'(g(1))g'(1) = f'(3)\,g'(1) = (15)(5) = \boxed{75}$$

8. (4 points) What is the difference between an explicitly-defined function and an implicitly-defined function? Give an example of each.

IF A FUNCTION IS EXPLICITLY DEFINED, THE DEPENDENT VARIABLE IS EXPLICITLY SOLVED FOR AND APPEARS BY ITSELF ON ONE SIDE OF AN EQUATION : $y = \sin x$.

IF IMPLICITLY DEFINED, THE FUNCTION IS DEFINED BY AN EQUATION WHERE THE DEPENDENT VARIABLE IS NOT ISOLATED:

$$x^2y + y^3 = 12x$$

9. (4 points) Use the quotient rule to derive the formula for the derivative of $y = \cot x$.

$$\frac{d}{dx} \cot x = \frac{d}{dx} \frac{\cos x}{\sin x} = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x} = -\csc^2 x$$

10. (7 points) Find the absolute extreme values of $f(x) = x^3 + 3x^2 + 1$ on $[-1, 2]$.

$f'(x) = 3x^2 + 6x$

ENDPOINTS: $x = -1,\ x = 2$

$f'(x)$ DNE NEVER

$f'(x) = 0$

$\Rightarrow 3x^2 + 6x = 0$

$3x(x+2) = 0$

$x = 0,\ x = -2$

$\boxed{x = 0,\ y = f(0) = 1}$ ← ABS MIN

$x = -1,\ y = f(-1) = 3$

$\boxed{x = 2,\ y = f(2) = 21}$ ← ABS MAX

11. (5 points) Determine $\dfrac{d^3}{dx^3}\sqrt{x}$.

$y = x^{1/2},\quad \dfrac{dy}{dx} = \dfrac{1}{2}x^{-1/2}$

$$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-3/2}$$

$$\boxed{\frac{d^3y}{dx^3} = \frac{3}{8}x^{-5/2}}$$

12. (9 points) Gravel is being dumped from a conveyor belt onto a conical pile in such a way that the volume of the pile is increasing at a rate of 30 ft^3/min. As the gravel pile grows, the base diameter and the height of the conical pile are always equal. How fast is the height of the pile increasing at the moment when the pile is 10 ft tall?

Variables: V = VOLUME AT TIME t

r = RADIUS OF BASE AT TIME t

h = HEIGHT AT TIME t

Given information and what to find:

$$\frac{dV}{dt} = 30 \quad \text{FIND } \frac{dh}{dt} \text{ WHEN } h = 10$$

Equation(s) relating the variables:

$$\underbrace{V = \frac{1}{3}\pi r^2 h, \quad h = 2r}_{V = \frac{2}{3}\pi r^3} = \text{BASE DIAMETER}$$

Equation(s) relating the rates:

$$\underbrace{\frac{dV}{dt} = 2\pi r^2 \frac{dr}{dt}, \quad \frac{dh}{dt} = 2\frac{dr}{dt}}_{\frac{dV}{dt} = \pi r^2 \frac{dh}{dt}}$$

Solution:

WHEN $h = 10$,

$$r = 5 \Rightarrow 30 = \pi (5)^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{30}{25\pi} \text{ FT/MIN}$$

Follow-up question: Does the height of the pile grow more quickly when the pile is small or when the pile is big? Explain your reasoning.

$$\frac{dh}{dt} = \frac{dV/dt}{\pi r^2}$$

HEIGHT GROWS MORE QUICKLY WHEN PILE IS SMALL.

$\frac{dh}{dt}$ VARIES INVERSELY WITH r^2.