Math 171 - Final Exam

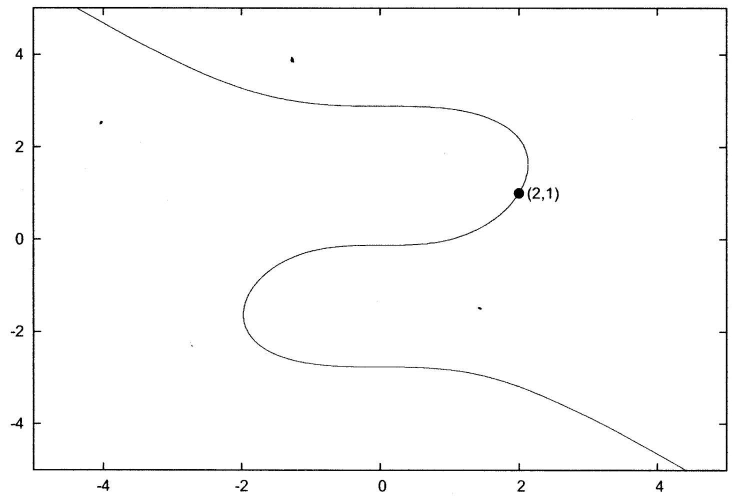
December 10, 2012

Name key

Score

Show all work to receive full credit. Supply explanations where necessary.

1. (8 points) Find an equation of the line tangent to the graph of the equation $x^3 + y^3 = 8y + 1$ at the point $(x, y) = (2, 1)$.



$$\frac{d}{dx}\left(x^3+y^3\right) = \frac{d}{dx}\left(8y+1\right)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 8\frac{dy}{dx}$$

$$\left(3y^2-8\right)\frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = \frac{-3x^2}{3y^2-8}$$

$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{-12}{-5} = \frac{12}{5}$$

$$\boxed{\text{TANGENT LINE: } y-1 = \frac{12}{5}(x-2)}$$

2. (8 points) Find the absolute maximum and minimum values of $g(x) = \frac{x}{2} + \cos x$ on the interval $[0, 2]$.

$$g'(x) = \frac{1}{2} - \sin x$$

$$g'(x) = 0 \Rightarrow \sin x = \frac{1}{2}$$

$$\Rightarrow x = \frac{\pi}{6}$$

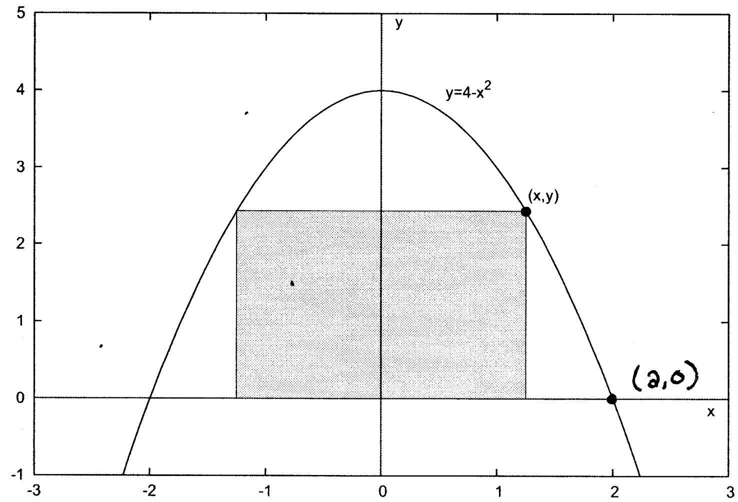
ONLY SOLUTION ON $[0,2]$

$g\left(\frac{\pi}{6}\right) \approx 1.1278 \leftarrow$ ABS MAX

$g(0) = 1$

$g(2) \approx 0.5839 \leftarrow$ ABS MIN

3. (12 points) A rectangle is bounded by the x-axis and the graph of $y = 4 - x^2$ (see below). Find the coordinates of the point (x, y) that maximize the area of the rectangle.



MAXIMIZE $A = xy$

SUBJECT TO $y = 4 - x^2$

$A(x) = 4x - x^3$

$A'(x) = 4 - 3x^2 = 0$

$\Rightarrow \quad x = \frac{2}{\sqrt{3}}.$

MAXIMIZE $A(x) = x(4 - x^2)$ ON $[0, 2]$

$A(0) = 0$

$A(2) = 0$

$A\left(\frac{2}{\sqrt{3}}\right) \approx 3.079 \leftarrow$ ABS MAX

$\boxed{(x, y) = \left(\frac{2}{\sqrt{3}}, \frac{8}{3}\right)}$

4. (8 points) Evaluate the definite integral: $\displaystyle\int_0^1 x^3 \cos(x^4 + 2)\, dx.$

$u = x^4 + 2$

$du = 4x^3\, dx$

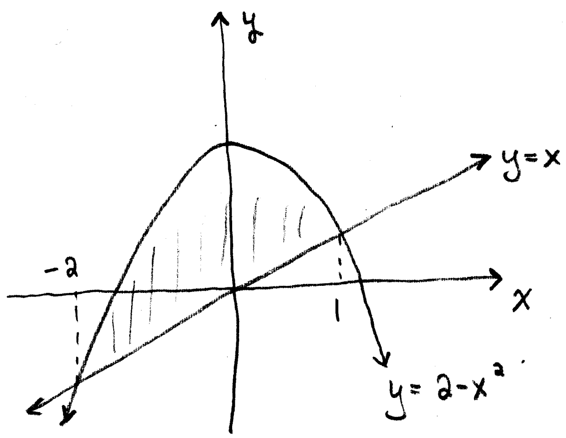
$\frac{1}{4}\, du = x^3\, dx$

$x = 0 \Rightarrow u = 2$

$x = 1 \Rightarrow u = 3$

$\frac{1}{4}\int_2^3 \cos u\, du = \frac{1}{4}\sin u \Big|_2^3 = \boxed{\frac{1}{4}(\sin 3 - \sin 2) \approx -0.192}$

5. (10 points) Find the area of the bounded region between the graphs of $y = 2 - x^2$ and $y = x$.



$$\text{Area} = \int_{-2}^{1} (2 - x^2 - x)\,dx$$

$$= 2x - \frac{1}{3}x^3 - \frac{1}{2}x^2 \Big|_{-2}^{1}$$

$$= \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - 2\right)$$

$$= \boxed{4.5}$$

$2 - x^2 = x$

$x^2 + x - 2 = 0$

$(x-1)(x+2) = 0$

$x = 1,\ x = -2$

6. (9 points) A person standing at the top of the Tower of Pisa throws a small heavy object directly upward so that after t seconds the object's height in feet is given by

$$s(t) = -16t^2 + 96t + 176.$$

(a) When does the object reach its maximum height?

$s'(t) = -32t + 96$

$s'(t) = 0 \Rightarrow \boxed{t = 3}$

AFTER 3 SECONDS.

(b) What is the maximum height of the object?

$s(3) = -16(9) + 96(3) + 176 = \boxed{320 \text{ FT}}$

(c) What is the velocity of the object when its height is 64 ft? What is the speed?

$s(t) = 64 \Rightarrow -16t^2 + 96t + 176 = 64$

$-16t^2 + 96t + 112 = 0$

$-16(t^2 - 6t - 7) = 0$

$-16(t-7)(t+1) = 0$

$t = 7$

$s'(7) = -32(7) + 96 = -128$

$$\boxed{\begin{array}{l} \text{VELOCITY} = -128 \text{ FT/SEC} \\ \text{SPEED} = 128 \text{ FT/SEC} \end{array}}$$

7. (16 points) Consider the function $f(x) = 2x(x-3)^2$.

(a) Find all critical numbers of f.

$$f'(x) = 2(x-3)^2 + 2x(2)(x-3)$$
$$= (x-3)\left[2x-6+4x\right]$$
$$= (x-3)(6x-6)$$

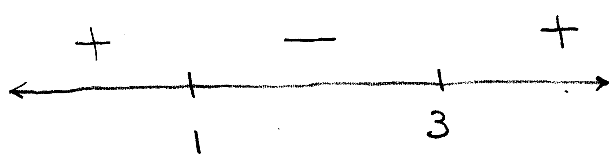
$f'(x)$ DNE NEVER

$f'(x) = 0$

$\Rightarrow \boxed{x = 3,\ x = 1}$

(b) Find open intervals on which f is increasing/decreasing.

SIGNS OF f'



INCREASING ON $(-\infty, 1) \cup (3, \infty)$

DECREASING ON $(1, 3)$

(c) Find the relative extreme values of f.

$f(1) = 8$ IS A RELATIVE MAX

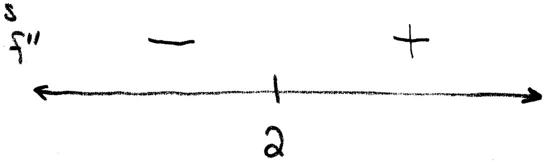
$f(3) = 0$ IS A RELATIVE MIN

(d) Find open intervals on which the graph of f is concave up/down.

$$f''(x) = (6x-6) + 6(x-3)$$
$$= 12x - 24$$

$f''(x) = 0 \Rightarrow x = 2$

SIGNS OF f''



CU ON $(2, \infty)$

CD ON $(-\infty, 2)$

8. (10 points) An oil tanker has run aground and ruptured its hull. Leaking oil is spreading in all directions. The polluted region is circular and growing steadily at a rate $15\,\text{m}^2/\text{hr}$. How fast is the radius of the oil slick growing at the moment when the radius is $25\,\text{m}$?

$A = \pi r^2$

A = AREA OF SPILL AT TIME t

r = RADIUS OF SPILL AT TIME t

FIND $\frac{dr}{dt}$ WHEN $r = 25$.

$\frac{dA}{dt} = 15$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

WHEN $r = 25$...

$15 = 2\pi(25)\frac{dr}{dt}$

$$\boxed{\frac{dr}{dt} = \frac{15}{50\pi} \text{ m/hr}}$$

9. (12 points) Find $\dfrac{dy}{dx}$. Do not simplify.

(a) $y = 3\sqrt{1-4x^2} = 3(1-4x^2)^{1/2}$

$$\boxed{\frac{dy}{dx} = \frac{3}{2}(1-4x^2)^{-1/2}(-8x)}$$

(b) $y = \dfrac{\sec x}{x^2+1}$

$$\boxed{\frac{dy}{dx} = \frac{(x^2+1)(\sec x \tan x) - (\sec x)(2x)}{(x^2+1)^2}}$$

(c) $y = x^3 \sin 2x$

$$\boxed{\frac{dy}{dx} = x^3(2)(\cos 2x) + 3x^2 \sin 2x}$$

10. (12 points) Find each limit analytically. Use ∞, $-\infty$, or DNE if appropriate.

(a) $\lim_{x \to 3} \frac{\frac{1}{x} - \frac{1}{3}}{x-3}$

$\frac{0}{0} \Rightarrow$ more work

$$\lim_{x \to 3} \frac{\frac{3-x}{3x}}{x-3} = \lim_{x \to 3} \frac{1}{3x}\left(\frac{3-x}{x-3}\right) = -\lim_{x \to 3} \frac{1}{3x}$$

$$= \boxed{-\frac{1}{9}}$$

(b) $\lim_{y \to 4^+} \frac{y^2 - 3y - 4}{3y - 12}$.

$\frac{0}{0} \Rightarrow$ more. work

$$\lim_{y \to 4+} \frac{(y-4)(y+1)}{3(y-4)} = \lim_{y \to 4+} \frac{y+1}{3} = \boxed{\frac{5}{3}}$$

(c) $\lim_{x \to -\infty} \frac{4 - 7x - 8x^2}{4x^2 + 6x + 2} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$$= \lim_{x \to -\infty} \frac{\frac{4}{x^2} - \frac{7}{x} - 8}{4 + \frac{6}{x} + \frac{2}{x^2}} = \frac{0 - 0 - 8}{4 + 0 + 0} = \boxed{-2}$$

11. (5 points) Find the linearization of $f(x) = \tan x$ at $x = \pi/4$.

$f'(x) = \sec^2 x$

$f'(\frac{\pi}{4}) = 2$

$f(\frac{\pi}{4}) = 1$

$L(x) = f(\frac{\pi}{4}) + f'(\frac{\pi}{4})(x - \frac{\pi}{4})$

$$\boxed{L(x) = 1 + 2(x - \frac{\pi}{4})}$$

12. (8 points) Evaluate the indefinite integral: $\int \left(5x\sqrt{x} + \frac{2}{x^5} + \sin x\right) dx$

$$\int (5x^{3/2} + 2x^{-5} + \sin x)\, dx$$

$$= \boxed{2x^{5/2} - \frac{1}{2}x^{-4} - \cos x + C}$$

13. (8 points) Consider the definite integral $\int_0^1 \frac{1}{x^2+1}\,dx$.

(a) Briefly explain how we can be sure that the value of this integral is positive.

$f(x) = \frac{1}{x^2+1}$ IS A POSITIVE CONTINUOUS FUNCTION ON $[0,1]$

(b) Use the trapezoid rule with $n = 4$ to approximate the value of the integral.

$h = \frac{1}{4}$, $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{2}{4}$, $x_3 = \frac{3}{4}$, $x_4 = \frac{4}{4}$

$$\frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{1}{8}\left[1 + \frac{2}{1.0625} + \frac{2}{1.25} + \frac{2}{1.5625} + \frac{1}{2}\right]$$

$$\approx \boxed{0.7828}$$

14. (6 points) Let $y = \frac{25}{x^2}$. Compute the differential dy and use it to approximate the change in y as x changes from $\underbrace{5 \text{ to } 5.03}$.

$y = 25x^{-2}$ $\quad dy = \frac{-50}{x^3}\,dx$ $\quad \Delta x = 0.03$ $\quad \Delta y \approx -\frac{50}{x^3}\Delta x$

$$\Delta y \approx \frac{-50}{5^3}(0.03) = \boxed{-\frac{3}{250} = -0.012}$$

15. (4 points) What is the difference between a removable discontinuity and a nonremovable discontinuity?

A DISCONTINUITY AT $x = c$ IS REMOVABLE AT $x = c$ IF $\lim_{x \to c} f(x)$ EXISTS.

OTHERWISE THE DISCONTINUITY IS NONREMOVABLE.

16. (14 points) Do any TWO of the following problems in the space provided below.

(a) Find the average value of $f(x) = 3x^5 - x^3 + x$ on the interval $[-3, 3]$.

(b) Use Newton's method to find the only positive solution of $x^2 = \sin x$.

(c) The area under the graph of $y = f(x)$ on the interval $[2, 4]$ is approximated by a Riemann sum of the form .

$$\sum_{k=1}^{n}(c_k^2 + c_k)\Delta x,$$

where the interval $[2, 4]$ is partitioned into n subintervals of equal width Δx and c_k is some point in the kth subinterval. Write and evaluate the definite integral that gives the exact area.

(d) Use the limit definition of the derivative to find $f'(x)$ if $f(x) = \sqrt{x-1}$.

(e) Compute the limit: $\lim_{\theta \to 0} \frac{\sin 5\theta}{3\theta}$

a) $\text{Avg VALUE} = \frac{1}{3-(-3)} \int_{-3}^{3} (3x^5 - x^3 + x)\,dx = 0$ SINCE f IS AN ODD FUNCTION ON $[-3,3]$.

b) $f(x) = x^2 - \sin x$

$f'(x) = 2x - \cos x$

X_0 = STARTING GUESS $= 1$

$X_{N+1} = X_N - \frac{f(X_N)}{f'(X_N)}$

$X_1 = 0.8913959953$

$X_2 = 0.8769848448$

$X_3 = 0.8767262985$

$X_4 = 0.8767262154$

X_5 = SAME AS X_4

c) $\int_2^4 (x^2 + x)\,dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 \Big|_2^4$

$= \left(\frac{64}{3} + 8\right) - \left(\frac{8}{3} + 2\right) = \boxed{\frac{74}{3}}$

d) $f'(x) = \lim_{h \to 0} \frac{\sqrt{(x+h)-1} - \sqrt{x-1}}{h} \cdot \frac{\sqrt{(x+h)-1} + \sqrt{x-1}}{\sqrt{(x+h)-1} + \sqrt{x-1}} = \lim_{h \to 0} \frac{(x+h)-1-(x-1)}{h\left(\sqrt{(x+h)-1} + \sqrt{x-1}\right)}$

$= \lim_{h \to 0} \frac{1}{\sqrt{(x+h)-1} + \sqrt{x-1}} = \boxed{\frac{1}{2\sqrt{x-1}}}$

e) $\lim_{\theta \to 0} \frac{\sin 5\theta}{3\theta} = \frac{1}{3} \lim_{\theta \to 0} \frac{5 \sin 5\theta}{5\theta} = \boxed{\frac{5}{3}}$