Math 171 - Final Exam

December 9, 2013

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary.

1. (10 points) Let $f(x) = \sqrt{x^2 + x + 4}$. Find an equation of the line tangent to the graph of f at the point where $x = 3$.

$f'(x) = \frac{1}{2}(x^2+x+4)^{-1/2}(2x+1)$

$f'(3) = \frac{1}{2}(16)^{-1/2}(7) = \frac{7}{8}$

$f(3) = \sqrt{16} = 4$

$\Rightarrow$ TAN LINE IS $y - 4 = \frac{7}{8}(x-3)$

2. (8 points) Find k so that g is continuous at $x = 2$.

$$g(x) = \begin{cases} 1 + \dfrac{3\sin(x-2)}{x-2}, & x < 2 \\ kx^2 + 3x - 5, & x \geq 2 \end{cases}$$

$\lim_{x \to 2^-} g(x) = g(2) \Rightarrow \lim_{x \to 2^-} \left(1 + \frac{3\sin(x-2)}{x-2}\right)$

$= 4k + 6 - 5$

$\Rightarrow 4 = 4k + 1$

$\Rightarrow \boxed{k = \frac{3}{4}}$

3. (8 points) Use the quotient rule to derive the formula for the derivative of $y = \tan x$.

$$\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x \quad \boxtimes$$

4. (16 points) Find each limit analytically. Use ∞, $-\infty$, or DNE if appropriate.

(a) $\lim\limits_{x\to\infty} \dfrac{5x^2 - 7x + 3}{8x^3 - 27}$ $\quad \dfrac{1/x^3}{1/x^3} = \lim\limits_{x\to\infty} \dfrac{\frac{5}{x} - \frac{7}{x^2} + \frac{3}{x^3}}{8 - \frac{27}{x^3}} = \dfrac{0}{8} = \boxed{0}$

(b) $\lim\limits_{x\to\pi} (x + \cos^2 x) = \pi + \cos^2 \pi = \pi + (-1)^2 = \boxed{\pi + 1}$

$0/0$

(c) $\lim\limits_{x\to 2} \dfrac{\sqrt{2x} - 2}{x^2 - 4} \cdot \dfrac{\sqrt{2x} + 2}{\sqrt{2x} + 2} = \lim\limits_{x\to 2} \dfrac{2x - 4}{(x^2-4)(\sqrt{2x}+2)} = \lim\limits_{x\to 2} \dfrac{2(x-2)}{(x-2)(x+2)(\sqrt{2x}+2)}$

$$= \lim_{x\to 2} \frac{2}{(x+2)(\sqrt{2x}+2)} = \frac{2}{(4)(4)} = \boxed{\frac{1}{8}}$$

$0/0$

(d) $\lim\limits_{x\to 0^+} \dfrac{3\sin 7x}{2x}$

$$= \lim_{x\to 0^+} \frac{3}{2} \, \frac{7 \sin 7x}{7x}^{\to 1} = \left(\frac{3}{2}\right)(7)(1) = \boxed{\frac{21}{2}}$$

5. (10 points) Let $f(x) = x^2 - 2x$. Use the limit definition of derivative to find $f'(x)$.

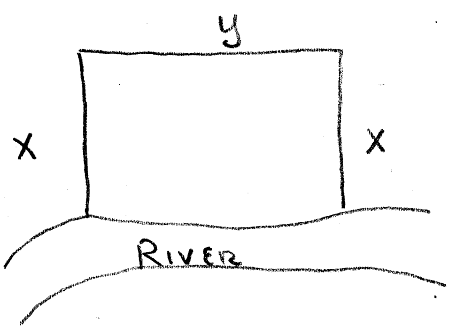
$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[(x+\Delta x)^2 - 2(x+\Delta x)\right] - \left[x^2 - 2x\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - 2x - 2\Delta x - x^2 + 2x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2 - 2\Delta x}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 2)$$

$$= \boxed{2x - 2}$$

6. (12 points) A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must have an area of 88,200 square meters. No fencing is needed along the river. What dimensions will require the least amount of fencing?



Minimize $P = 2x + y$

s.t. $xy = 88200$

$$y = \frac{88200}{x}$$

$$P(x) = 2x + \frac{88200}{x}, \quad x > 0$$

$$P'(x) = 2 - \frac{88200}{x^2} = 0 \Rightarrow x^2 = 44100$$

$$\Rightarrow x = 210$$

$$P''(x) = \frac{2(88200)}{x^3} \Rightarrow P''(210) > 0$$

$\Rightarrow x = 210$ gives A MIN

Dimensions are

$x = 210$ FT

$y = \frac{88200}{210} =$ 420 FT

7. (20 points) Consider the following rational functions:

$$f(x)\ \frac{x^2-9}{x^2-1}$$

and

$$f'(x) = \frac{16x}{(x^2-1)^2} \qquad f''(x) = \frac{-16(3x^2+1)}{(x^2-1)^3}.$$

(a) Find the horizontal and vertical asymptotes of the graph of f.

V. A.

INFINITE LIMITS WHEN $X = \pm 1$

H. A.

$\lim_{X \to \pm\infty} f(x) = 1$

$\boxed{\text{V. A.'s ARE } X = 1,\ X = -1 \quad \text{H. A. IS } y = 1}$

(b) Determine the intervals on which f is increasing/decreasing.

$f'(x) = 0 \Rightarrow X = 0$

$X = \pm 1$ NOT IN DOMAIN

SIGNS OF f': $-$ $-$ $+$ $+$

-1 $\quad 0 \quad$ 1

$\boxed{\begin{array}{l} f \text{ IS DEC ON } (-\infty,-1) \cup (-1,0) \\ f \text{ IS INC ON } (0,1) \cup (1,\infty) \end{array}}$

(c) Determine the intervals on which the graph of f is concave up/down.

$f''(x)$ IS NEVER ZERO

$X = \pm 1$ NOT IN DOMAIN

SIGNS OF f'': $-$ $+$ $-$

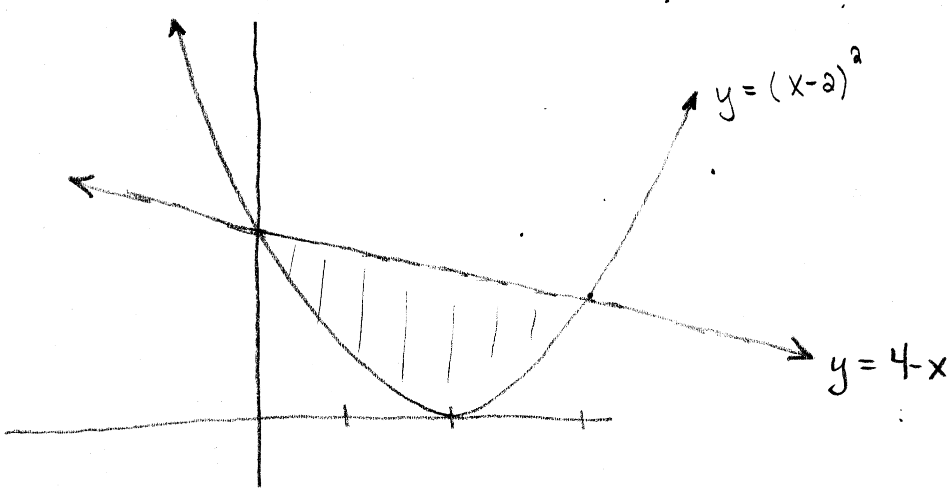
-1 $\quad 1$

$\boxed{\begin{array}{l} \text{GRAPH IS CU ON } (-1,1) \\ \text{GRAPH IS CD ON } (-\infty,-1) \cup (1,\infty) \end{array}}$

(d) Identify all relative extrema.

$\boxed{f(0) = 9 \text{ IS A RELATIVE MIN}}$

8. (12 points) Find the area of the region bounded by the graphs of $y = \overbrace{x^2 - 4x + 4}^{(x-2)^2}$ and $y = 4 - x$. You may use your calculator to evaluate your definite integral.



$x^2 - 4x + 4 = 4 - x$

$\Rightarrow x^2 - 3x = 0$

$x(x-3) = 0$

$x = 0,\ x = 3$

$$\text{Area} = \int_0^3 (4-x) - (x^2 - 4x + 4)\, dx$$

CALCULATOR SAYS ...

$$\boxed{\text{Area} = 4.5}$$

9. (8 points) Evaluate the definite integral: $\int_{-1}^{1} 5x^2 (x^3 + 1)^4\, dx$.

$u = x^3 + 1$

$du = 3x^2\, dx \qquad \frac{1}{3} du = x^2\, dx$

$x = -1 \Rightarrow u = 0$

$x = 1 \Rightarrow u = 2$

$$\frac{5}{3} \int_0^2 u^4\, du$$

$$= \frac{1}{3} u^5 \Big|_0^2 = \boxed{\frac{32}{3}}$$

10. (20 points) The height of a ball thrown vertically upward is given by

$$s(t) = -16t^2 + 96t + 112$$

where s is measured in feet and t is measured in seconds.

(a) What are the initial height and velocity of the ball?

INITIAL HEIGHT = 112 FT

INITIAL VELOCITY = 96 FT/S

(b) What is the maximum height attained by the ball?

$$S'(t) = -32t + 96 = 0 \Rightarrow t = 3$$

$$S(3) = \boxed{256 \text{ FT}}$$

(c) When does the ball hit the ground?

$$S(t) = 0 \Rightarrow -16(t^2 - 6t - 7) = -16(t-7)(t+1) = 0$$

$$\boxed{t = 7 \text{ s}}$$

(d) What is the speed of the ball when it hits the ground?

$$S'(7) = -32(7) + 96 = -128$$

$$\boxed{\text{Speed is } 128 \text{ FT/S}}$$

11. (10 points) Evaluate the indefinite integral: $\int \left(7x\sqrt{x^3} - \frac{6}{x^2} + \sec^2 x\right)\, dx$

$$\int \left(7x x^{3/2} - 6x^{-2} + \sec^2 x\right) dx = \int \left(7x^{5/2} - 6x^{-2} + \sec^2 x\right) dx$$

$$= \boxed{2x^{7/2} + 6x^{-1} + \tan x + C}$$

12. (8 points) Consider the definite integral $\int_0^1 \sin x^2\, dx$. Use the trapezoid rule with $n = 4$ to approximate the value of the integral.

$$T = \frac{1}{8}\Big(\sin(0) + 2\sin(0.25^2) + 2\sin(0.5^2) + 2\sin(0.75^2) + \sin(1)\Big)$$

$$= \frac{1}{8}(2.527802886\ldots)$$

$$\approx \boxed{0.316}$$

13. (8 points) Find the average value of $y = x + \sin x$ on the interval from $x = 0$ to $x = \pi$.

$$\frac{1}{\pi}\int_0^{\pi} (x + \sin x)\, dx$$

$$= \frac{1}{\pi}\left(\frac{1}{2}x^2 - \cos x\right)\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{1}{2}\pi^2 + 1\right) - (0 - 1)\right]$$

$$= \boxed{\frac{1}{\pi}\left(\frac{1}{2}\pi^2 + 2\right) \approx 2.207}$$