Math 172 - Test 2

October 18, 2017

Name key

Score ________

Show all work to receive full credit. Supply explanations where necessary. Unless otherwise stated, you may use your calculator to evaluate any **definite** integrals.

1. (5 points) Use the definitions of the hyperbolic functions (in terms of exponential functions) to prove the following identity:

$$\sinh 2x = 2 \sinh x \cosh x$$

$2 \sinh x \cosh x$

$$= (e^x - e^{-x})\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^{2x} - 1 + 1 - e^{-2x}}{2} = \frac{e^{2x} - e^{-2x}}{2}$$

$$= \sinh 2x$$

2. (5 points) Using the definitions of the hyperbolic functions (in terms of exponential functions) and the quotient rule, show that $\frac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x$.

$$\frac{d}{dx} \operatorname{sech} x = \frac{d}{dx} \frac{2}{e^x + e^{-x}} = \frac{-2(e^x - e^{-x})}{(e^x + e^{-x})^2} = -\frac{2}{e^x + e^{-x}}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$$

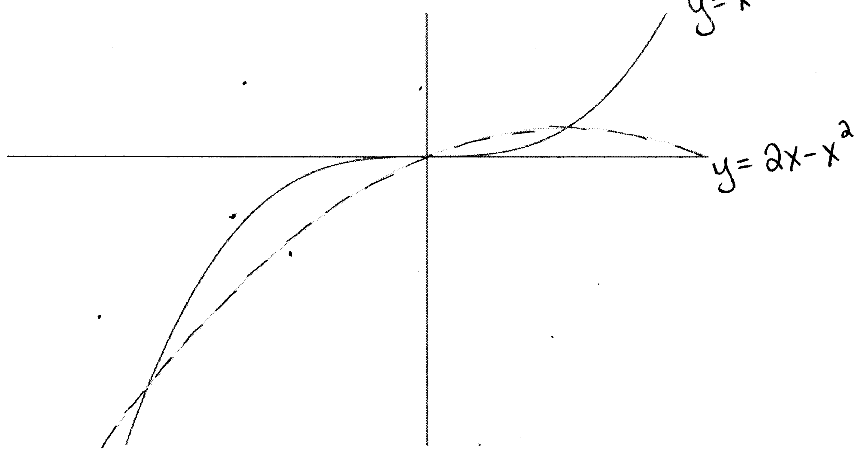
$$= -\operatorname{sech} x \tanh x$$

3. (5 points) Use the derivative formula, $\frac{d}{dx}[\tanh^{-1} x] = \frac{1}{1 - x^2}$, to determine the derivative of $y = \tanh^{-1} \sqrt{x}$.

$$\frac{dy}{dx} = \frac{1}{1 - (\sqrt{x})^2} \frac{d}{dx} \sqrt{x} = \frac{1}{1 - x}\left(\frac{1}{2} x^{-1/2}\right)$$

$$= \frac{1}{2\sqrt{x}(1 - x)}$$

4. (15 points) Consider the region between the graphs of $y = x^3$ and $y = 2x - x^2$. The graphs are shown below.



(a) Find the x-coordinates of the points at which the graphs intersect.

$$x^3 = 2x - x^2 \Rightarrow x^3 + x^2 - 2x = 0$$

$$x(x+2)(x-1) = 0$$

$$x = 0, \quad x = -2, \quad x = 1$$

(b) Find the area of the combined regions enclosed by the graphs. **Evaluate your definite integral(s) by hand.**

$$\int_{-2}^{0} (x^3 - 2x + x^2)\,dx \quad + \quad \int_0^1 (2x - x^2 - x^3)\,dx$$

$$\left(\frac{1}{4}x^4 - x^2 + \frac{1}{3}x^3\right)\Big|_{-2}^{0} \quad + \quad \left(x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^1$$

$$-\left(4 - 4 - \frac{8}{3}\right) \quad + \left(1 - \frac{1}{3} - \frac{1}{4}\right)$$

$$\frac{8}{3} \quad + \quad \frac{5}{12} \quad = \quad \boxed{\frac{37}{12}}$$

5. (10 points) Find the length of the graph of $f(x) = \ln x - \dfrac{x^2}{8}$ from $x = 1$ to $x = 2$. **Evaluate your definite integral by hand.** Then use your calculator to evaluate the integral to check your work.

$$f'(x) = \frac{1}{x} - \frac{x}{4}$$

$$ds = \sqrt{1 + \left(\frac{1}{x} - \frac{x}{4}\right)^2}\,dx$$

$$= \sqrt{1 + \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{16}}\,dx = \sqrt{\left(\frac{1}{x} + \frac{x}{4}\right)^2}\,dx$$

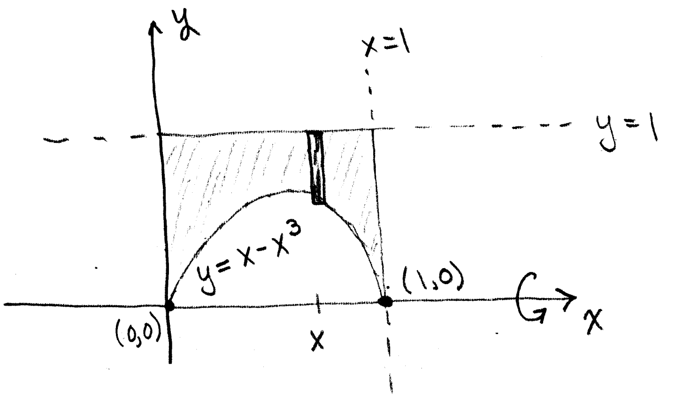
$$= \left(\frac{1}{x} + \frac{x}{4}\right)dx$$

$$\int_1^2 \left(\frac{1}{x} + \frac{x}{4}\right)dx = \left(\ln x + \frac{x^2}{8}\right)\Big|_1^2$$

$$= \ln 2 + \frac{3}{8} \approx \boxed{1.068}$$

$$\text{fnInt}\left(\sqrt{\left(1 + (1/x - x/4)^2\right)}, x, 1, 2\right) \approx 1.068$$

6. (8 points) The 1st quadrant region bounded by the graphs $y = x - x^3$, $x = 1$, and $y = 1$ is rotated about the x-axis to generate a solid of revolution. Find the volume of the solid. Use your calculator to evaluate the required integral(s).

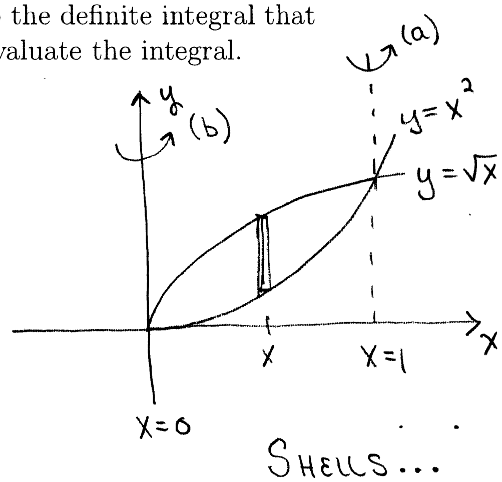


WASHERS...

$$\text{Volume} = \pi \int_0^1 1^2 - (x - x^3)^2\,dx$$

$$= \frac{97\pi}{105} \approx 2.9$$

7. (12 points) The region bounded by the graphs of $f(x) = \sqrt{x}$ and $y = x^2$ is rotated about the given line to generate a solid of revolution. Set up the definite integral that gives the volume of the solid. Then use your calculator to evaluate the integral.



(a) $x = 1$

$$2\pi \int_0^1 (1-x)(\sqrt{x} - x^2)\,dx$$

$$= \frac{11\pi}{30} \approx 1.15$$

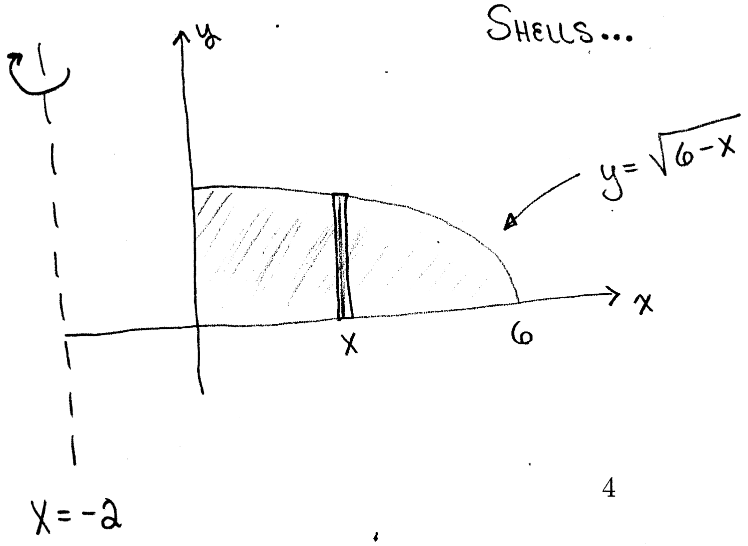
(b) $x = 0$

$$2\pi \int_0^1 x(\sqrt{x} - x^2)\,dx$$

$$= \frac{3\pi}{10} \approx 0.94$$

8. (6 points) The expression $2\pi \int_0^6 (x+2)\sqrt{6-x}\,dx$ gives the volume of a solid of revolution. Identify the plane region being rotated and the axis of revolution.

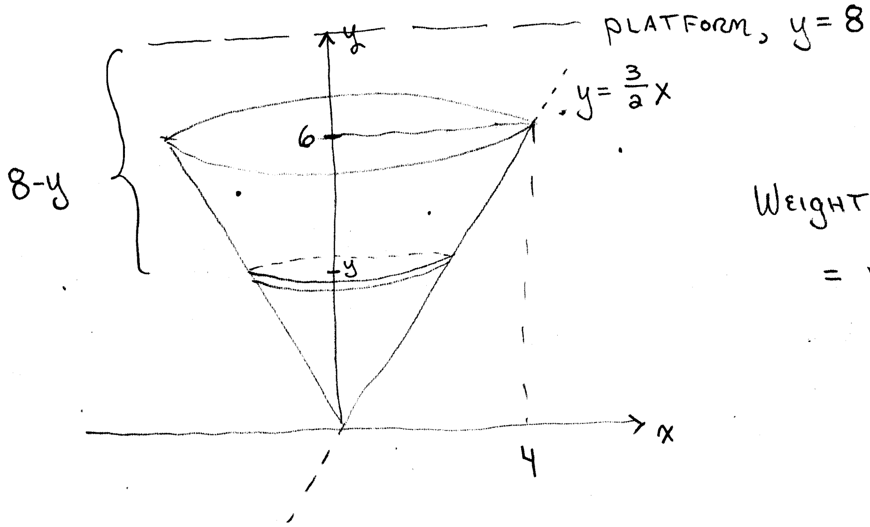
SHELLS...



1ST QUAD REGION BOUNDED BY $y=\sqrt{6-x}$, $x=0$, $y=0$ ABOUT $x=-2$

FILLED WITH WATER

9. (14 points) Water weighs 62.4 pounds per cubic foot. An open tank has the shape of a right circular cone with its point down. The tank is 8 feet across the top and 6 feet high. How much work is done in emptying the tank by pumping the water to a platform 2 feet above the top edge of the tank?



WEIGHT OF DISK AT y

$= \text{VOLUME} \times 62.4$

$= \pi \left(\frac{2}{3}y\right)^2 dy \cdot 62.4$

$$\text{WORK} = \int_0^6 62.4\pi \left(\frac{2}{3}y\right)^2 (8-y)\, dy$$

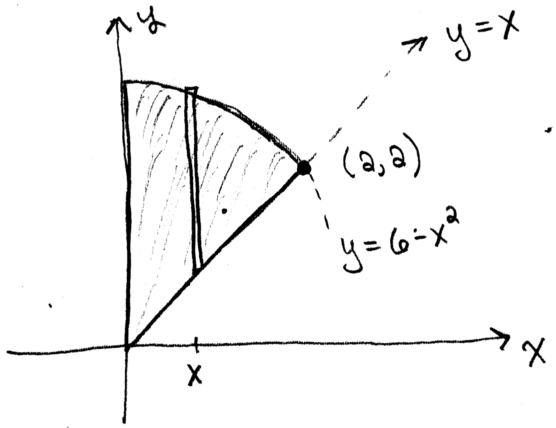
$$= \frac{(62.4)\pi(4)}{9} \int_0^6 (8y^2 - y^3)\, dy$$

$$= \frac{(62.4)\pi(4)}{9} \left(\frac{8}{3}y^3 - \frac{1}{4}y^4\right)\Big|_0^6$$

$$= \frac{(62.4)\pi(4)}{9} (576 - 324)$$

$$\approx \boxed{21955.96 \text{ ft-lb}}$$

10. (20 points) A thin plate is lying in the 1st quadrant bounded by the graphs of $y = 6 - x^2$, $y = x$, and $x = 0$. The density of the plate at the point (x, y) is given by $\rho(x) = 1 + 2x$. Find the center of mass of the plate. You may use your calculator to evaluate the required integrals.



X=0

$$dm = (1+2x)(6-x^2-x)\,dx$$

$6-x^2 = x$

$x^2+x-6=0$

$(x+3)(x-2)=0$

$x=2$

$$\text{Mass} = \int_0^2 (1+2x)(6-x^2-x)\,dx = 18$$

$$M_y = \int_0^2 x(1+2x)(6-x^2-x)\,dx = \frac{248}{15}$$

$$M_x = \int_0^2 \left(\frac{6-x^2+x}{2}\right)(1+2x)(6-x^2-x)\,dx = \frac{788}{15}$$

$$\text{C.M.} = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{124}{135}, \frac{394}{135}\right)$$

$$\approx (0.92, 2.92)$$