

Math 172 - Final Exam A
December 11, 2017

Name key Score _____

Show all work to receive full credit. Supply explanations where necessary. YOU MUST WORK INDIVIDUALLY ON THIS EXAM.

1. (5 points) Evaluate: $\lim_{x \rightarrow \infty} x \left(e^{\frac{1}{2x}} - e^{\frac{1}{4x}} \right)$

$$\lim_{x \rightarrow \infty} \frac{e^{\frac{1}{2x}} - e^{\frac{1}{4x}}}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{2} \left(-\frac{1}{x^2} \right) e^{\frac{1}{2x}} - \frac{1}{4} \left(-\frac{1}{x^2} \right) e^{\frac{1}{4x}}}{\left(-\frac{1}{x^2} \right)}$$

$$= \lim_{x \rightarrow \infty} \left(\frac{1}{2} e^{\frac{1}{2x}} - \frac{1}{4} e^{\frac{1}{4x}} \right) = \frac{1}{2} - \frac{1}{4}$$

$$= \boxed{\frac{1}{4}}$$

2. (5 points) Evaluate: $\int_0^5 \frac{1}{25-x^2} dx = \int_0^5 \left(\frac{1}{5-x} - \frac{1}{5+x} \right) dx$

$$\frac{1}{(5-x)(5+x)} = \frac{A}{5-x} + \frac{B}{5+x}$$

$$1 = A(5+x) + B(5-x)$$

$$x=5: 1 = 10A \Rightarrow A = \frac{1}{10}$$

$$x=-5: 1 = -10B \Rightarrow B = -\frac{1}{10}$$

$$= \frac{1}{10} \int_0^5 \left(\frac{1}{5-x} - \frac{1}{5+x} \right) dx$$

①

$$\int_0^5 \frac{1}{5+x} dx$$

$$= \ln(5+x) \Big|_0^5$$

$$= \ln 10 - \ln 5$$

$$\textcircled{2} \quad \int_0^5 \frac{1}{5-x} dx = \lim_{c \rightarrow 5^-} \int_0^c \frac{1}{5-x} dx$$

$$= \lim_{c \rightarrow 5^-} -\ln|5-x| \Big|_0^c$$

$$= \lim_{c \rightarrow 5^-} (-\ln(5-c))$$

$$+ \ln 5$$

$$= \infty$$

INTEGRAL DIVERGES

3. (a) (3 points) Determine the second Maclaurin polynomial for $f(x) = e^x$.

$$P_2(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2$$

$$f(0) = 1$$

$$f'(x) = e^x, \quad f'(0) = 1$$

$$f''(x) = e^x, \quad f''(0) = 1$$

$$P_2(x) = 1 + x + \frac{x^2}{2}$$

- (b) (2 points) Now that you have a Maclaurin polynomial for $f(x) = e^x$, you can approximate $e^{\frac{1}{kx}}$ by substituting $\frac{1}{kx}$ in place of the x in your Maclaurin polynomial. Re-evaluate the limit in Problem #1 by using this idea.

$$e^{\frac{1}{kx}} \approx 1 + \frac{1}{kx} + \frac{1}{2k^2x^2} \Rightarrow e^{\frac{1}{2x}} \approx 1 + \frac{1}{2x} + \frac{1}{8x^2} \quad \left. \begin{array}{l} e^{\frac{1}{2x}} - e^{\frac{1}{4x}} \\ e^{\frac{1}{4x}} \approx 1 + \frac{1}{4x} + \frac{1}{32x^2} \end{array} \right\} \approx \frac{1}{4x} + \frac{3}{32x^2}$$

$$\lim_{x \rightarrow \infty} x(e^{\frac{1}{2x}} - e^{\frac{1}{4x}}) = \lim_{x \rightarrow \infty} x \left(\frac{1}{4x} + \frac{3}{32x^2} \right) = \lim_{x \rightarrow \infty} \left(\frac{1}{4} + \frac{3}{32x} \right) = \frac{1}{4}$$

4. (6 points) Determine the third Maclaurin polynomial for $g(x) = \tan x$. Then use it to approximate $\tan(\pi/30)$.

$$g(0) = 0$$

$$g'(x) = \sec^2 x, \quad g'(0) = 1$$

$$g''(x) = 2 \underbrace{\sec x \sec x}_{\sec^2 x} \tan x, \quad g''(0) = 0$$

$$g'''(x) = 4 \underbrace{\sec x \sec x \tan x}_{\sec^2 x} + 2 \sec^2 x \sec^2 x, \quad g'''(0) = 2$$

$$P_3(x) = g(0) + g'(0)(x-0) + \frac{g''(0)}{2}(x-0)^2 + \frac{g'''(0)}{6}(x-0)^3$$

$$P_3(x) = x + \frac{2}{6}x^3 = x + \frac{x^3}{3}$$

$$\tan\left(\frac{\pi}{30}\right) \approx \frac{\pi}{30} + \frac{(\pi/30)^3}{3}$$

2

$$\approx 0.105103$$

ACTUAL VALUE OF $\tan\left(\frac{\pi}{30}\right)$ IS ABOUT
0.105104

$$\rightarrow \frac{3\ln n}{6n} = \frac{\ln n}{2n}$$

5. (3 points) The n th term of a sequence is given by $a_n = \frac{\ln(n^3)}{6n}$. Write the first four terms of the sequence. Then determine if the sequence converges or diverges.

$$\frac{\ln 1}{2}, \frac{\ln 2}{4}, \frac{\ln 3}{6}, \frac{\ln 4}{8}$$

$$\lim_{x \rightarrow \infty} \frac{\ln x}{2x} \stackrel{\infty/\infty}{=} \lim_{x \rightarrow \infty} \frac{1}{2x} = 0 \Rightarrow \text{SEQUENCE CONVERGES TO ZERO}$$

6. (8 points) Consider the series $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$.

- (a) Use the logarithm laws to rewrite the terms of the series.

$$\sum_{n=1}^{\infty} \left[\ln n - \ln(n+1) \right]$$

- (b) Find a formula for the n th partial sum of the series.

$$S_1 = \ln 1 - \ln 2$$

$$S_2 = (\ln 1 - \ln 2) + (\ln 2 - \ln 3)$$

⋮

$$S_n = \ln 1 - \ln(n+1)$$

$$S_n = -\ln(n+1)$$

- (c) Does the series converge or diverge? Explain how you know.

$$\text{BECAUSE } \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} [-\ln(n+1)] \\ = -\infty$$

- (d) Explain why the n th term test does not apply to this series.

$$\text{BECAUSE } \lim_{n \rightarrow \infty} \ln\left(\frac{n}{n+1}\right) = 0.$$

n^{TH} TERM TEST ONLY SAYS WHAT HAPPENS
WHEN LIMIT IS NOT ZERO.

7. (18 points) Determine whether each series converges conditionally, converges absolutely, or diverges.

$$(a) \sum_{n=1}^{\infty} \frac{4n}{2n^2+1}$$

Compare with $\sum \frac{1}{n}$ which diverges (Harmonic Series)

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{4n}{2n^2+1}} = \lim_{n \rightarrow \infty} \frac{2n^2+1}{4n^2} = \frac{1}{2}$$

Series Diverges by limit comparison.

$$(b) \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \quad (\text{Hint: Use ratio test.})$$

$$\lim_{n \rightarrow \infty} \left| \frac{[(n+1)!]^2}{(2n+2)!} \cdot \frac{(2n)!}{[n!]^2} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{4n^2 + 6n + 2} = \frac{1}{4} < 1$$

Series CONVERGES
(ABSOLUTELY).

$$(c) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+1)}$$

$$\frac{1}{\ln(n+2)} < \frac{1}{\ln(n+1)} \Rightarrow \text{TERMS DECREASE}$$

$$\lim_{n \rightarrow \infty} \frac{1}{\ln(n+1)} = 0$$

} Series CONVERGES
By AST.

However,

SINCE

$$\frac{1}{\ln(n+1)} > \frac{1}{n} \text{ AND } \sum \frac{1}{n} \text{ DIVERGES,}$$

THE SERIES DOES NOT

CONVERGE ABSOLUTELY

Convergence
IS
CONDITIONAL.

Math 172 - Final Exam B

December 11, 2017

Name key Score _____

Show all work to receive full credit. Supply explanations where necessary. All integrals must be evaluated by hand, with work shown, unless otherwise stated.

1. (10 points) For $x \geq 1$, let $f(x) = \frac{100x\sqrt{x-1}}{(2x+1)^2}$. Determine $f'(x)$ and then find an equation of the line tangent to the graph of f at $x = 2$. (Hint: Use logarithmic differentiation.)

$$\ln f(x) = \ln 100 + \ln x + \frac{1}{2} \ln(x-1) - 2 \ln(2x+1)$$

$$\frac{1}{f(x)} f'(x) = \frac{1}{x} + \frac{1}{2(x-1)} - \frac{4}{2x+1}$$

$$f'(x) = \left(\frac{100x\sqrt{x-1}}{(2x+1)^2} \right) \left(\frac{1}{x} + \frac{1}{2(x-1)} - \frac{4}{2x+1} \right)$$

$$m = f'(2) = \left(\frac{200}{25} \right) \left(\frac{1}{2} + \frac{1}{2} - \frac{4}{5} \right) = \frac{8}{5}$$

TAN LINE:

$$y - 8 = \frac{8}{5}(x-2)$$

$\uparrow f(2)$

2. (8 points) Let $f(x) = x^5 + 7x + 2$.

- (a) Briefly explain how we can be certain that f has an inverse.

$$f'(x) = 5x^4 + 7 > 0 \text{ for all } x \Rightarrow f \text{ is increasing} \Rightarrow f \text{ is 1-1}$$

- (b) Briefly explain why it would be difficult to find a formula for $f^{-1}(x)$.

DIFFICULT OR IMPOSSIBLE TO SOLVE

$$y = x^5 + 7x + 2$$

FOR x .

- (c) Compute $(f^{-1})'(10)$.

$$f'(x) = 5x^4 + 7$$

$$f^{-1}(10) = u \Leftrightarrow f(u) = 10$$

$$\Leftrightarrow u = 1$$

$$(f^{-1})'(10) = \frac{1}{f'(f^{-1}(10))}$$

$$= \frac{1}{5+7} = \boxed{\frac{1}{12}}$$

3. (6 points) Evaluate the indefinite integral: $\int x \sec x \tan x dx$

$$u = x$$

$$du = dx$$

$$dv = \sec x \tan x dx \quad v = \sec x$$

$$x \sec x - \int \sec x dx$$

$$= x \sec x - \ln |\sec x + \tan x| + C.$$

4. (12 points) Use a partial fraction decomposition to evaluate the following indefinite integral.

$$\int \frac{5x^2 + 3x + 22}{(x^2 + 9)(x - 1)} dx$$

$$\frac{5x^2 + 3x + 22}{(x^2 + 9)(x - 1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + 9}$$

$$5x^2 + 3x + 22 = A(x^2 + 9) + (Bx + C)(x - 1)$$

$$x = 1 : 30 = 10A \Rightarrow A = 3$$

$$x = 0 : 22 = 9A - C \Rightarrow 22 = 27 - C \Rightarrow C = 5$$

$$x = -1 : 24 = 10A + (C - B)(-2)$$

$$\Rightarrow 24 = 30 - 10 + 2B \Rightarrow B = 2$$

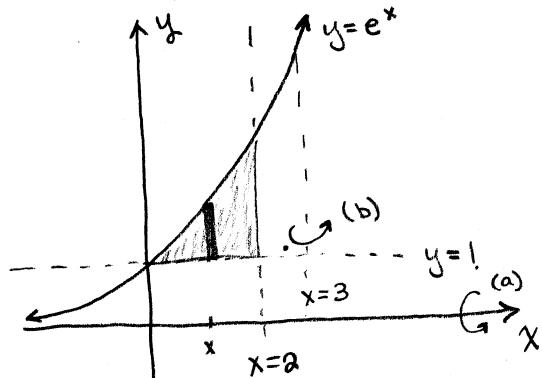
$$\int \left(\frac{3}{x-1} + \frac{2x}{x^2 + 9} + \frac{5}{x^2 + 9} \right) dx = 3 \ln |x-1| + \ln (x^2 + 9) + \frac{5}{3} \tan^{-1} \frac{x}{3} + C$$

$$\begin{matrix} u = x^2 + 9 \\ du = 2x dx \end{matrix}$$

$$\int \frac{1}{u} du$$

5. Consider the 1st-quadrant region bounded by the graphs of $y = e^x$, $y = 1$, and $x = 2$.

- (a) (8 points) The region is rotated about the x -axis to generate a solid of revolution. Find the volume of the solid. Evaluate the definite integral(s) by hand.



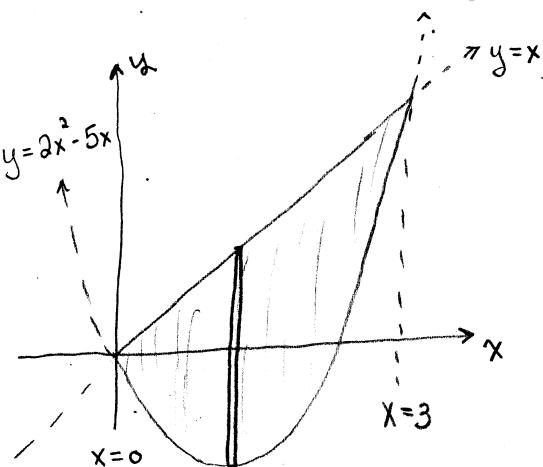
$$\begin{aligned}
 & \text{WASHERS...} \\
 \pi \int_0^2 (e^x)^2 - (1)^2 dx &= \pi \int_0^2 (e^{2x} - 1) dx \\
 &= \pi \left[\frac{1}{2} e^{2x} - x \right]_0^2 = \pi \left[\frac{e^4}{2} - 2 - \frac{1}{2} \right] \\
 &= \boxed{\frac{(e^4 - 5)\pi}{2}} \approx 77.9
 \end{aligned}$$

- (b) (5 points) The region is rotated about the line $x = 3$ to generate a solid of revolution. Find the volume of the solid. Use your calculator to evaluate the definite integral(s).

SHells...

$$\begin{aligned}
 2\pi \int_0^2 (3-x)(e^x - 1) dx &= 2\pi(2e^2 - 8) \\
 &\approx 42.6
 \end{aligned}$$

6. (10 points) A thin plate covers the region between the graphs of $y = 2x^2 - 5x$ and $y = x$. The density of the plate at the point (x, y) is given by $\delta(x) = x$. Set up the integral that gives the plate's moment about the x -axis. Use your calculator to evaluate the integral.



$$\begin{aligned}
 M_x &= \int_0^3 y dm \\
 &= \int_0^3 \underbrace{\left(\frac{x + 2x^2 - 5x}{2} \right)}_{y} (x) \underbrace{(x - 2x^2 + 5x)}_{\text{DENSITY} \times \text{AREA}} dx \\
 &= \boxed{0}
 \end{aligned}$$

$$\begin{aligned}
 2x^2 - 5x &= x \\
 \Rightarrow 2x^2 - 6x &= 0 \Rightarrow 2x(x-3) = 0
 \end{aligned}$$

$0 \cdot (-\infty)$

7. (5 points) Evaluate the limit: $\lim_{x \rightarrow 0^+} x \ln x$

$$\begin{aligned}
 & \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x}} \quad -\infty/\infty \\
 & \text{L'Hôpital's Rule} \\
 & = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 0^+} -x^2 = \boxed{0}
 \end{aligned}$$

8. (6 points) Write the following improper integral as a limit of proper integrals and evaluate. (Hint: You'll need your result from above.)

$$\int_0^1 \ln x \, dx$$

$$\int \ln x \, dx = x \ln x - \int 1 \, dx = x \ln x - x$$

$$u = \ln x \quad du = \frac{1}{x} dx$$

$$dv = dx \quad v = x$$

$$\begin{aligned}
 & \lim_{a \rightarrow 0^+} \int_a^1 \ln x \, dx = \lim_{a \rightarrow 0^+} \left(x \ln x - x \Big|_a^1 \right) \\
 & = \lim_{a \rightarrow 0^+} \left(-1 - a \ln a + a \right) = \boxed{-1}
 \end{aligned}$$

9. (10 points) Use an appropriate trigonometric substitution to evaluate the following definite integral.

$$\int_0^1 \frac{1}{(x^2 + 1)^{3/2}} dx$$

$$x = \tan \theta$$

$$x = 0 \Rightarrow \theta = 0$$

$$dx = \sec^2 \theta d\theta$$

$$x = 1 \Rightarrow \theta = \frac{\pi}{4}$$

$$\begin{aligned} \int_0^{\pi/4} \frac{\sec^2 \theta}{(\tan^2 \theta + 1)^{3/2}} d\theta &= \int_0^{\pi/4} \frac{\sec^2 \theta}{|\sec \theta|^3} d\theta && \text{on } [0, \pi/4], \\ &= \int_0^{\pi/4} \frac{1}{\sec \theta} d\theta = \int_0^{\pi/4} \cos \theta d\theta \\ &= \sin \theta \Big|_0^{\pi/4} = \boxed{\frac{\sqrt{2}}{2}} \end{aligned}$$

10. (5 points) Consider the series $\sum_{n=1}^{\infty} \frac{8}{3^n}$.

- (a) Find the fourth partial sum of the series.

$$\frac{8}{3} + \frac{8}{9} + \frac{8}{27} + \frac{8}{81} = \boxed{\frac{320}{81} \approx 3.95}$$

- (b) Does the series converge or diverge? Explain how you know.

Geometric with $r = \frac{1}{3}$

\Rightarrow Series converges.

- (c) If the series converges, find its sum.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{8}{3^n} &= \sum_{n=0}^{\infty} \frac{8}{3^n} - 8 \\ &= \frac{8}{1 - \frac{1}{3}} - 8 = \boxed{4} \end{aligned}$$

11. (15 points) Determine whether each series converges or diverges. Show all work and explain your reasoning.

$$(a) \sum_{n=1}^{\infty} \frac{2^n}{n!}$$

RATIO TEST:

$$\lim_{n \rightarrow \infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n \rightarrow \infty} \frac{n! \cdot 2^{n+1}}{(n+1)! \cdot 2^n} = \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0 < 1$$

Series converges.

$$(b) \sum_{n=1}^{\infty} \ln \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \ln \frac{1}{n} = -\infty$$

Series diverges

By n^{th} term test.

$$(c) \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - n}$$

$$\text{Look at } \sum_{n=2}^{\infty} \frac{1}{n^2 - n}.$$

Limit comparison with $\sum \frac{1}{n^2}$, which converges ($p=2$).

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{n^2 - n}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2 - n} = 1 \Rightarrow \sum \frac{1}{n^2 - n} \text{ converges.}$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - n} \text{ converges (ABSOLUTELY)}$$